Section 5: Guided Electromagnetic Waves

5.1 Wave guides

So far, we have dealt with plane waves of infinite extent; now we consider electromagnetic waves confined to the interior of a hollow pipe, or *wave guide* (Fig. 5.1). We assume the wave guide is a perfect conductor, so that **E** = 0 and **B** = 0 inside the material itself, and hence the boundary conditions at the inner wall are

$$\mathbf{E}^{\parallel} = 0 \ , \tag{5.1}$$

$$B^{\perp} = 0 \ . \tag{5.2}$$

The latter condition follows from the fact that in a perfect conductor, **E** = 0, and hence (by Faraday's law) $\partial\mathbf{B}/\partial t = 0$; assuming the magnetic field started out zero, then, it will remain so.

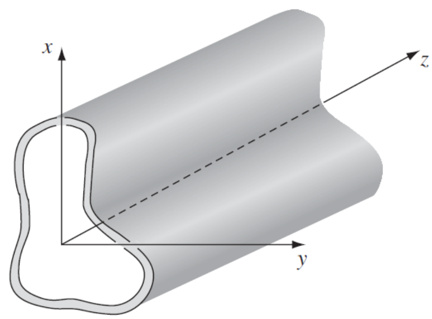

Fig. 5.1

Free charges and currents are induced on the surface in such a way as to enforce these constraints. We are interested in monochromatic waves that propagate down the tube, so **E** and **B** have the generic form

$$\mathbf{E}(x,y,z,t) = \mathbf{E}_0(x,y)e^{i(kz-\omega t)} \ , \tag{5.3}$$

$$\mathbf{B}(x,y,z,t) = \mathbf{B}_0(x,y)e^{i(kz-\omega t)} \ . \tag{5.4}$$

The electric and magnetic fields must satisfy Maxwell's equations, in the interior of the wave guide:

$$\nabla \cdot \mathbf{E} = 0 \, , \tag{5.5}$$

$$\nabla \cdot \mathbf{B} = 0 \, , \tag{5.6}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \, , \tag{5.7}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \ . \tag{5.8}$$

The problem, then, is to find functions $\mathbf{E}_0$ and $\mathbf{B}_0$ such that the fields (5.3) and (5.4) obey the differential equations (5.5)-(5.8), subject to boundary conditions (5.1), (5.2).

As we will see below, confined waves are not (in general) transverse. Therefore, in order to satisfy the boundary conditions, we have to include longitudinal (z) components:

$$\mathbf{E}_0 = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}} \, , \tag{5.9}$$

$$\mathbf{B}_0 = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}} \, , \tag{5.10}$$

where each of the components is a function of x and y. Putting this into Maxwell's equations (5.7) and (5.8), we obtain

$$\left(\nabla\times\mathbf{E}\right)_x=\left(\frac{\partial E_z}{\partial y}-ikE_y\right)e^{i(kz-\omega t)} \quad\Rightarrow\quad \frac{\partial E_z}{\partial y}-ikE_y=i\omega B_x\,, \tag{5.11}$$

$$\left(\nabla\times\mathbf{E}\right)_y=\left(ikE_x-\frac{\partial E_z}{\partial x}\right)e^{i(kz-\omega t)} \quad\Rightarrow\quad ikE_x-\frac{\partial E_z}{\partial x}=i\omega B_y\,, \tag{5.12}$$

$$\left(\nabla\times\mathbf{E}\right)_z=\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)e^{i(kz-\omega t)} \quad\Rightarrow\quad \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega B_z\,, \tag{5.13}$$

$$\left(\nabla\times\mathbf{B}\right)_x=\left(\frac{\partial B_z}{\partial y}-ikB_y\right)e^{i(kz-\omega t)} \quad\Rightarrow\quad \frac{\partial B_z}{\partial y}-ikB_y=-\frac{i\omega}{c^2}E_x\,, \tag{5.14}$$

$$\left(\nabla\times\mathbf{B}\right)_y=\left(ikB_x-\frac{\partial B_z}{\partial x}\right)e^{i(kz-\omega t)} \quad\Rightarrow\quad ikB_x-\frac{\partial B_z}{\partial x}=-\frac{i\omega}{c^2}E_y\,, \tag{5.15}$$

$$\left(\nabla\times\mathbf{B}\right)_z=\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)e^{i(kz-\omega t)} \quad\Rightarrow\quad \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=-\frac{i\omega}{c^2}E_z\,. \tag{5.16}$$

Equations (5.11), (5.12), (5.14), and (5.15) can be solved for E_x, E_y, B_x, and B_y, which can be expressed in terms of z components of the fields. Multiplying (5.12) by k and (5.14) by ω and subtracting, we find:

$$ik^2E_x-k\frac{\partial E_z}{\partial x}-\omega\frac{\partial B_z}{\partial y}+ik\omega B_y=i\omega kB_y+\frac{i\omega^2}{c^2}E_x \quad\Rightarrow\quad E_x=i\left(\frac{\omega^2}{c^2}-k^2\right)^{-1}\left(k\frac{\partial E_z}{\partial x}+\omega\frac{\partial B_z}{\partial y}\right). \tag{5.17}$$

Multiplying (5.11) by k and (5.15) by ω and adding, we find:

$$ik\omega B_x-\omega\frac{\partial B_z}{\partial x}+k\frac{\partial E_z}{\partial y}-ik^2E_y=i\omega kB_x-\frac{i\omega^2}{c^2}E_y \quad\Rightarrow\quad E_y=i\left(\frac{\omega^2}{c^2}-k^2\right)^{-1}\left(k\frac{\partial E_z}{\partial y}-\omega\frac{\partial B_z}{\partial x}\right). \tag{5.18}$$

Multiplying (5.11) by ω/c^2 and (5.15) by k and adding we find:

$$ik^2B_x-k\frac{\partial B_z}{\partial x}+\frac{\omega}{c^2}\frac{\partial E_z}{\partial y}-\frac{i\omega k}{c^2}E_y=i\omega\frac{\omega}{c^2}B_x-\frac{i\omega k}{c^2}E_y \quad\Rightarrow\quad B_x=i\left(\frac{\omega^2}{c^2}-k^2\right)^{-1}\left(k\frac{\partial B_z}{\partial x}-\frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right). \tag{5.19}$$

Multiplying (5.12) by ω/c^2 and (5.14) by k and subtracting we find:

$$ik\frac{\omega}{c^2}E_x-\frac{\omega}{c^2}\frac{\partial E_z}{\partial x}-k\frac{\partial B_z}{\partial y}+ik^2B_y=i\omega\frac{\omega}{c^2}B_y+i\frac{\omega}{c^2}kE_x \quad\Rightarrow\quad B_y=i\left(\frac{\omega^2}{c^2}-k^2\right)^{-1}\left(\frac{\omega}{c^2}\frac{\partial E_z}{\partial x}+k\frac{\partial B_z}{\partial y}\right). \tag{5.20}$$

It suffices, then, to determine the longitudinal components E_z and B_z; if we knew those, we could quickly calculate the others, just by differentiating. To find E_z and B_z, we use the remaining Maxwell's equations:

$$\nabla\cdot\mathbf{E}=\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+ikE_z\right)e^{i(kz-\omega t)}=0 \quad\Rightarrow\quad \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+ikE_z=0\,. \tag{5.21}$$

Substituting Eqs. (5.17) and (5.18) we obtain

$$i\left(\frac{\omega^2}{c^2}-k^2\right)^{-1}\left(k\frac{\partial^2 E_z}{\partial x^2}+\omega\frac{\partial^2 B_z}{\partial x\partial y}\right)+i\left(\frac{\omega^2}{c^2}-k^2\right)^{-1}\left(k\frac{\partial^2 E_z}{\partial y^2}-\omega\frac{\partial^2 B_z}{\partial x\partial y}\right)+ikE_z=0 \tag{5.22}$$

or

$$\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+\left(\frac{\omega^2}{c^2}-k^2\right)E_z=0. \tag{5.23}$$

Likewise,

$$\nabla\cdot\mathbf{B}=\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+ikB_z\right)e^{i(kz-\omega t)}=0 \quad\Rightarrow\quad \frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+ikB_z=0 \tag{5.24}$$

Substituting Eqs. (5.19) and (5.20) we obtain

$$i\left(\frac{\omega^2}{c^2}-k^2\right)^{-1}\left(k\frac{\partial^2 B_z}{\partial x^2}-\frac{\omega}{c^2}\frac{\partial^2 E_z}{\partial x\partial y}\right)+i\left(\frac{\omega^2}{c^2}-k^2\right)^{-1}\left(\frac{\omega}{c^2}\frac{\partial^2 E_z}{\partial x\partial y}+k\frac{\partial^2 B_z}{\partial y^2}\right)+ikB_z=0 \tag{5.25}$$

or

$$\frac{\partial^2 B_z}{\partial x^2}+\frac{\partial^2 B_z}{\partial y^2}+\left(\frac{\omega^2}{c^2}-k^2\right)B_z=0. \tag{5.26}$$

If $E_z = 0$ these waves are called **TE** ("transverse electric") waves. If $B_z = 0$ they are called **TM** ("transverse magnetic") waves. If both $E_z = 0$ and $B_z = 0$, we call them **TEM** waves. It turns out that TEM waves cannot occur in a hollow wave guide.

Proof: If $E_z = 0$, Gauss's law (5.5) says

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=0, \tag{5.27}$$

and if $B_z = 0$, Faraday's law (5.7) says

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=0. \tag{5.28}$$

Indeed, the vector $\mathbf{E}_0$ in Eq. (5.9) has zero divergence and zero curl. It can therefore be written as the gradient of a scalar potential that satisfies Laplace's equation. But the boundary condition on **E** requires that the surface be an equipotential, and since Laplace's equation admits no local maxima or minima, this means that the potential is constant throughout, and hence the electric field is *zero* – no wave at all. Notice that this argument applies only to a completely *empty* pipe – if you ran a separate conductor down the middle, the potential at *its* surface need not be the same as on the outer wall, and hence a nontrivial potential is possible.

Let us now summarize the results. The solution of the Maxwell equations for a hollow waveguide represents propagating waves in the z direction ($\sim e^{i(kz-\omega t)}$). The amplitudes of these waves can be found from differential equations (5.17)-(5.20), (5.23), (5.26) which we rewrite in the following form:

$$E_x=\frac{i}{\gamma^2}\left(k\frac{\partial E_z}{\partial x}+\omega\frac{\partial B_z}{\partial y}\right), \tag{5.29}$$

$$E_y=\frac{i}{\gamma^2}\left(k\frac{\partial E_z}{\partial y}-\omega\frac{\partial B_z}{\partial x}\right), \tag{5.30}$$

$$B_x = \frac{i}{\gamma^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y} \right), \tag{5.31}$$

$$B_y = \frac{i}{\gamma^2}\left(\frac{\omega}{c^2}\frac{\partial E_z}{\partial x} + k\frac{\partial B_z}{\partial y} \right), \tag{5.32}$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \gamma^2 E_z = 0\,, \tag{5.33}$$

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \gamma^2 B_z = 0\,, \tag{5.34}$$

where

$$\gamma^2 \equiv \frac{\omega^2}{c^2} - k^2\,. \tag{5.35}$$

There are two linear independent solutions of these equations corresponding to TM and TE waves. For TM waves B_z = 0 everywhere. The solution of equation (5.33) should be found subject to boundary condition $\mathbf{E}^{\parallel} = 0$ which implies that on the surface

$$E_z\big|_S = 0\,, \tag{5.36}$$

for TE waves E_z = 0 everywhere. The solution of equation (5.34) should be obtained subject to boundary condition $B^{\perp} = \mathbf{B}\cdot\mathbf{n} = 0$, where $\mathbf{n} = (n_x, n_y)$ is the normal to the surface *S*. It follows from equations (5.31) and (5.32) that this boundary condition implies $\frac{\partial B_z}{\partial x}n_x + \frac{\partial B_z}{\partial y}n_y = \nabla B_z \cdot \mathbf{n} = 0$. The latter equation can be written as follows:

$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0\,. \tag{5.37}$$

Equations (5.33) and (5.34), together with boundary conditions (5.36) and (5.37) specify an eigenvalue problem. The constant γ^2 must be nonnegative, otherwise the wave will decay exponentially with *z*. There will be a spectrum of eigenvalues γ_λ^2 and corresponding solutions for λ = 1, 2, 3, ..., which form an orthogonal set. These different solutions are called the *modes of the guide.* For a given frequency ω, the wave number *k* is determined for each value of λ:

$$k_\lambda^2 = \frac{\omega^2}{c^2} - \gamma_\lambda^2\,. \tag{5.38}$$

If we define a *cutoff frequency*

$$\omega_\lambda = c\gamma_\lambda\,, \tag{5.39}$$

then the wave vector can be written as

$$k_\lambda = \frac{1}{c}\sqrt{\omega^2 - \omega_\lambda^2}\,. \tag{5.40}$$

We note that, for $\omega > \omega_\lambda$, the wave number k_λ is real; waves of the λ mode can propagate in the guide. For frequencies less than the cutoff frequency, k_λ is imaginary; such modes cannot propagate and are called cutoff modes or *evanescent* modes. The behavior of the axial wave number as a function of frequency is shown qualitatively in Fig. 5.2. We see that at any given frequency only a finite number of modes can propagate. It is often convenient to choose the dimensions of the guide so that at the operating frequency only the lowest mode can occur. This is shown by the vertical arrow on the figure.

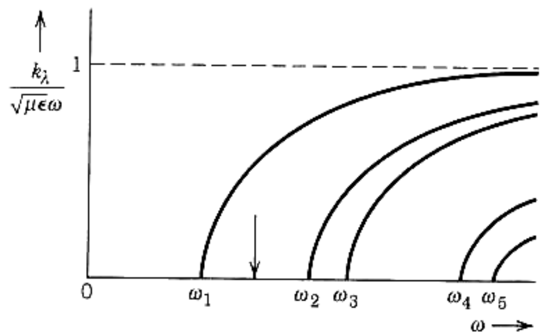

Fig. 5.2. Wave number k_λ versus frequency ω for various modes λ. ω_λ is the cutoff frequency.

5.2 Waves in a Rectangular Wave Guide

Suppose we have a wave guide of rectangular shape (Fig. 5.3) with height *a* and width *b,* and we are interested in the propagation of TE waves. The problem is to solve Eq. (5.34), subject to the boundary condition (5.37). We solve it by separation of variables. Let

$$B_z(x, y) = X(x)Y(y) \tag{5.41}$$

so that

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k^2\right)XY = 0. \tag{5.42}$$

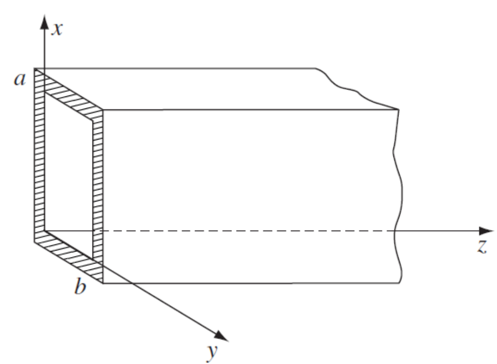

Fig. 5.3

Divide by *XY* and note that the *x*- and *y*-dependent terms must be constant:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2, \tag{5.43}$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2, \tag{5.44}$$

with

$$-k_x^2 - k_y^2 + \frac{\omega^2}{c^2} - k^2 = 0. \tag{5.45}$$

The general solution is

$$X(x) = A\sin(k_x x) + B\cos(k_x x). \tag{5.46}$$

But the boundary conditions (5.37) require that $dX/dx = 0$ at $x = 0$ and $x = a$. So $A = 0$, and

$$k_x = m\pi / a, \quad m = 0, 1, 2\ldots. \tag{5.47}$$

The same goes for Y, with

$$k_y = n\pi / b, \quad n = 0, 1, 2\ldots, \tag{5.48}$$

and we conclude that

$$B_z(x, y) = B_0 \cos(m\pi x / a)\cos(n\pi y / b). \tag{5.49}$$

This solution is called the TE_{mn} mode. The first index is conventionally associated with the *larger* dimension, so we assume $a > b$. At least *one* of the indices must be nonzero. The wave number k is given by

$$k^2 = \frac{\omega^2}{c^2} - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]. \tag{5.50}$$

The cutoff frequency is

$$\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}. \tag{5.51}$$

If $\omega < \omega_{mn}$ the wave number is imaginary, and instead of a traveling wave we have exponentially attenuated fields. The *lowest* cutoff frequency for a given wave guide occurs for the mode

$$\omega_{10} = \frac{c\pi}{a}; \tag{5.52}$$

frequencies less than this will not propagate at all.

The wave number can be written more simply in terms of the cutoff frequency:

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}. \tag{5.53}$$

The *phase* velocity is

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \omega_{mn}^2 / \omega^2}}. \tag{5.54}$$

which is greater than c. However, as we know, the energy carried by the wave travels at the *group* velocity

$$v_g = \frac{1}{dk / d\omega} = c\sqrt{1 - \omega_{mn}^2 / \omega^2}, \tag{5.55}$$

which is less than c.

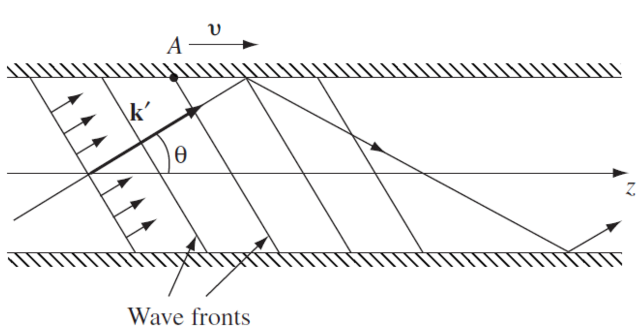

Fig. 5.4

There is another way to visualize the propagation of an electromagnetic wave in a rectangular pipe, and it serves to illuminate many of these results. Consider an ordinary *plane* wave, traveling at an angle θ to the z axis, and reflecting perfectly off each conducting surface (Fig. 5.4). In the x and y directions the (multiply reflected) waves interfere to form standing wave patterns, of wavelength $\lambda_x = 2a/m$ and $\lambda_y = 2b/n$ (hence wave number $k_x = 2\pi/\lambda_x = \pi m/a$ and $k_y = 2\pi/\lambda_y = \pi n/b$), respectively. Meanwhile, in the z direction there remains a traveling wave, with wave number $k_z = k$. The propagation vector for the "original" plane wave is therefore

$$\mathbf{k}' = \frac{\pi m}{a}\hat{\mathbf{x}} + \frac{\pi n}{b}\hat{\mathbf{y}} + k\hat{\mathbf{z}} . \tag{5.56}$$

and the frequency is

$$\omega = c\left|\mathbf{k}'\right| = c\sqrt{\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2 + k^2} = \sqrt{c^2k^2 + \omega_{mn}^2} . \tag{5.57}$$

Only certain angles will lead to one of the allowed standing wave patterns:

$$\cos\theta = \frac{k}{\left|\mathbf{k}'\right|} = \sqrt{1 - \omega_{mn}^2 / \omega^2} . \tag{5.58}$$

The plane wave travels at speed c, but because it is going at an angle θ to the z axis, its *group* velocity down the wave guide is

$$v_g = c\cos\theta = c\sqrt{1 - \omega_{mn}^2 / \omega^2} , \tag{5.59}$$

The *phase* velocity, on the other hand, is the speed of the wave fronts (A, say, in Fig. 5.4) down the pipe. Like the intersection of a line of breakers with the beach, they can move much faster than the waves themselves – in fact

$$v = \frac{c}{\cos\theta} = \frac{c}{\sqrt{1 - \omega_{mn}^2 / \omega^2}} . \tag{5.60}$$

5.3 Coaxial Transmission Line

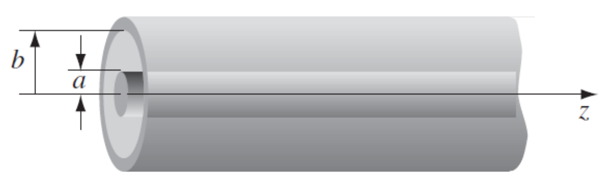

Fig. 5.5

We showed above that a *hollow* wave guide cannot support TEM waves. But a coaxial transmission line, consisting of a long straight wire of radius a, surrounded by a cylindrical conducting sheath of radius b (Fig. 5.5), *does* admit modes with $E_z = 0$ and $B_z = 0$. In this case Maxwell's equations (5.11) – (5.16)

$$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x , \tag{5.61}$$

$$ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y , \tag{5.62}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z , \tag{5.63}$$

$$\frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x \,, \tag{5.64}$$

$$ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y \,, \tag{5.65}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z \,, \tag{5.66}$$

take the form

$$-E_y = \frac{\omega}{k}B_x \,, \tag{5.67}$$

$$E_x = \frac{\omega}{k}B_y \,, \tag{5.68}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \,, \tag{5.69}$$

$$B_y = \frac{\omega}{k}\frac{1}{c^2}E_x \,, \tag{5.70}$$

$$B_x = -\frac{\omega}{k}\frac{1}{c^2}E_y \,, \tag{5.71}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0 \,. \tag{5.72}$$

From Eqs. (5.67) and (5.71), (5.68) and (5.70), we see that

$$c = \frac{\omega}{k} \,, \tag{5.73}$$

i.e. the wave travel at speed c and is nondispersive. We also see that

$$E_x = cB_y \,, \tag{5.74}$$

$$E_y = -cB_x \,, \tag{5.75}$$

so that $\mathbf{E}\cdot\mathbf{B} = 0$, i.e. **E** and **B** are mutually perpendicular. Eqs.(5.69) and (5.72) together with $\nabla\cdot\mathbf{E} = 0$ and $\nabla\cdot\mathbf{B} = 0$ are

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \,, \tag{5.76}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0 \,, \tag{5.77}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \,, \tag{5.78}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \,. \tag{5.79}$$

These are precisely the equations of *electrostatics* and *magnetostatics,* for empty space, in two dimensions;

the solution with cylindrical symmetry can be borrowed directly from the case of an infinite line charge and an infinite straight current, respectively:

$$\mathbf{E}(s,\phi)=\frac{E_0}{s}\hat{\mathbf{s}}\,, \tag{5.80}$$

$$\mathbf{B}(s,\phi)=\frac{E_0}{cs}\hat{\boldsymbol{\phi}}\,, \tag{5.81}$$

where E_0 is some constant. Substituting these into Eqs. (5.3) and (5.4), and taking the real part we find

$$\mathbf{E}(x,y,z,t)=\frac{E_0}{s}\cos(kz-\omega t)\hat{\mathbf{s}}\,, \tag{5.82}$$

$$\mathbf{B}(x,y,z,t)=\frac{E_0}{cs}\cos(kz-\omega t)\hat{\boldsymbol{\phi}}\,. \tag{5.83}$$

5.4 Resonant Cavities

A resonant cavity represents a waveguide with end faces placed on its length. Such a cavity can serve as an electromagnetic wave resonator to select or amplify waves at certain frequencies. In general, the electromagnetic waves in resonant cavities are standing waves due to reflections from end surfaces in all the three directions.

As an example, we consider the resonant cavity produced by closing off the two ends of a rectangular wave guide (Fig. 5.3), at $z = 0$ and at $z = d$, making a perfectly conducting empty box.

We look at the solutions of Maxwell's equations in the form of $\mathbf{E}(\mathbf{r},t)=\mathbf{E}(\mathbf{r})e^{-i\omega t}$, $\mathbf{B}(\mathbf{r},t)=\mathbf{B}(\mathbf{r})e^{-i\omega t}$, subject the boundary conditions $\mathbf{E}^{\|}=0$, $B^{\perp}=0$. Substituting to Eqs. (5.5)-(5.8), we obtain

$$\nabla\cdot\mathbf{E}=0\,, \tag{5.84}$$

$$\nabla\cdot\mathbf{B}=0\,, \tag{5.85}$$

$$\nabla\times\mathbf{E}=i\omega\mathbf{B}\,, \tag{5.86}$$

$$\nabla\times\mathbf{B}=-\frac{i\omega}{c^2}\mathbf{E}\,. \tag{5.87}$$

To find the electric field **E,** we take curl of Eq. (5.86) :

$$\nabla\times\left(\nabla\times\mathbf{E}\right)=\nabla\left(\nabla\cdot\mathbf{E}\right)-\nabla^2\mathbf{E}=-\nabla^2\mathbf{E}=\nabla\times\left(i\omega\mathbf{B}\right)=i\omega\left(-\frac{i\omega}{c^2}\mathbf{E}\right)=\frac{\omega^2}{c^2}\mathbf{E}\,. \tag{5.88}$$

Therefore,

$$\nabla^2\mathbf{E}=-\frac{\omega^2}{c^2}\mathbf{E}\,. \tag{5.89}$$

These are three differential equations for the *x*-, *y*-, and *z*-components of the electric field **E** which we solve by separation of variables. Assuming that

$$E_x(x,y,z)=X(x)Y(y)Z(z)\;, \tag{5.90}$$

we obtain

$$YZ\frac{d^2X}{dx^2}+ZX\frac{d^2Y}{dy^2}+XY\frac{d^2Z}{dz^2}=-\frac{\omega^2}{c^2}XYZ\;\Rightarrow\;\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=-\frac{\omega^2}{c^2}\;. \tag{5.91}$$

Each term must be a constant, so

$$\frac{d^2X}{dx^2}=-k_x^2X,\quad \frac{d^2Y}{dy^2}=-k_y^2Y,\quad \frac{d^2Z}{dz^2}=-k_z^2Z\,, \tag{5.92}$$

with

$$k_x^2+k_y^2+k_z^2=-\frac{\omega^2}{c^2}\,. \tag{5.93}$$

The solution is

$$E_x(x,y,z)=\left[A_1\sin(k_xx)+C_1\cos(k_xx)\right]\left[A_2\sin(k_yy)+C_2\cos(k_yy)\right]\left[A_3\sin(k_zz)+C_3\cos(k_zz)\right]\,. \tag{5.94}$$

But $\mathbf{E}^{\|}=0$ at the boundaries which means that $E_x=0$ at $y=0$ and $z=0$, so $C_2=C_2=0$, and $E_x=0$ at $y=b$ and $z=d$, so $k_y=n\pi/b$ and $k_z=l\pi/d$, where n and l are integers. Therefore, we obtain

$$E_x(x,y,z)=\left[A_1\sin(k_xx)+C_1\cos(k_xx)\right]\sin(k_yy)\sin(k_zz)\,. \tag{5.95}$$

Using similar arguments for y-, and z-components of $\mathbf{E}$, we obtain

$$E_y(x,y,z)=\sin(k_xx)\left[A_2\sin(k_yy)+C_2\cos(k_yy)\right]\sin(k_zz)\,, \tag{5.96}$$

$$E_z(x,y,z)=\sin(k_xx)\sin(k_yy)\left[A_3\sin(k_zz)+C_3\cos(k_zz)\right], \tag{5.97}$$

where $k_x=m\pi/a$. Actually, there is no reason at this point to assume that k_x, k_y and k_z are the same in all three components, but in a moment, we will see that in fact they do have to be the same.

From $\nabla\cdot\mathbf{E}=0$, we find

$$\begin{aligned}&k_x\left[A_1\cos(k_xx)-C_1\sin(k_xx)\right]\sin(k_yy)\sin(k_zz)+k_y\sin(k_xx)\left[A_2\cos(k_yy)-C_2\sin(k_yy)\right]\sin(k_zz)+\\&+k_z\sin(k_xx)\sin(k_yy)\left[A_3\cos(k_zz)-C_3\sin(k_zz)\right]=0.\end{aligned} \tag{5.98}$$

First, this equation tells us that k_x, k_y and k_z must be the same for different field components, because otherwise this equation could not be satisfied for all x, y, and z, due to linear independence of Fourier harmonics. Further, putting $x=0$ in Eq. (5.98), we obtain

$$k_xA_1\sin(k_yy)\sin(k_zz)=0\,, \tag{5.99}$$

and hence $A_1=0$. Likewise, putting $y=0$ in Eq. (5.98), we find $A_2=0$, and putting $z=0$, we find $A_3=0$. As the result, we obtain

$$C_1k_x+C_2k_y+C_3k_z=0\,, \tag{5.100}$$

and the electric field is

$$\mathbf{E}=C_1\cos(k_xx)\sin(k_yy)\sin(k_zz)\hat{\mathbf{x}}+C_2\sin(k_xx)\cos(k_yy)\sin(k_zz)\hat{\mathbf{y}}+C_3\sin(k_xx)\sin(k_yy)\cos(k_zz)\hat{\mathbf{z}}\,, \tag{5.101}$$

where $k_x=m\pi/a$, $k_y=n\pi/b$, and $k_z=l\pi/d$ (m, n, l are integers) and coefficients are related by Eq. (5.100).

The corresponding magnetic field $\mathbf{B}$ is given by $\mathbf{B}=-(i/\omega)\nabla\times\mathbf{E}$:

$$B_x = -\frac{i}{\omega}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = -\frac{i}{\omega}\Big[C_3 k_y \sin(k_x x)\cos(k_y y)\cos(k_z z) - C_2 k_z \sin(k_x x)\cos(k_y y)\cos(k_z z) \Big], \quad (5.102)$$

$$B_y = -\frac{i}{\omega}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) = -\frac{i}{\omega}\Big[C_1 k_z \cos(k_x x)\sin(k_y y)\cos(k_z z) - C_3 k_x \cos(k_x x)\sin(k_y y)\cos(k_z z) \Big], \quad (5.103)$$

$$B_z = -\frac{i}{\omega}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -\frac{i}{\omega}\Big[C_2 k_x \cos(k_x x)\cos(k_y y)\sin(k_z z) - C_1 k_y \cos(k_x x)\cos(k_y y)\sin(k_z z) \Big]. \quad (5.104)$$

This can be written more compact as follows:

$$\begin{aligned}\mathbf{B} = -\frac{i}{\omega}\Big[&\left(C_3 k_y - C_2 k_z\right)\sin(k_x x)\cos(k_y y)\cos(k_z z)\hat{\mathbf{x}} + \\ &+\left(C_1 k_z - C_3 k_x\right)\cos(k_x x)\sin(k_y y)\cos(k_z z)\hat{\mathbf{y}} + \left(C_2 k_x - C_1 k_y\right)\cos(k_x x)\cos(k_y y)\sin(k_z z)\hat{\mathbf{z}}\Big].\end{aligned} \quad (5.105)$$

This form of B automatically satisfies the boundary condition $B^{\perp} = 0$: $B_x = 0$ at $x = 0$ and $x = a$, $B_y = 0$ at $y = 0$ and $y = b$, and $B_z = 0$ at $z = 0$ and $z = d$.

Also, it is easy to see that $\nabla \cdot \mathbf{B} = 0$:

$$\begin{aligned}\nabla \cdot \mathbf{B} = -\frac{i}{\omega}\Big[&\left(C_3 k_y - C_2 k_z\right)k_x \cos(k_x x)\cos(k_y y)\cos(k_z z) + \\ &+\left(C_1 k_z - C_3 k_x\right)k_y \cos(k_x x)\cos(k_y y)\cos(k_z z) + \left(C_2 k_x - C_1 k_y\right)k_z \cos(k_x x)\cos(k_y y)\cos(k_z z)\Big] = \\ = -\frac{i}{\omega}&\left(C_3 k_y k_x - C_2 k_z k_x + C_1 k_z k_y - C_3 k_x k_y + C_2 k_x k_z - C_1 k_y k_z\right)\cos(k_x x)\cos(k_y y)\cos(k_z z) = 0.\end{aligned} \quad (5.106)$$

Thus, Eqs. (5.101) and (5.105) satisfy all Maxwell's equations and boundary conditions. For TE modes, we pick $E_z = 0$, so that $C_3 = 0$ and hence $C_1 k_x + C_2 k_y = 0$, leaving only the overall amplitude undetermined, for given *m*, *n*, and *l*. For TM modes $B_z = 0$, so $C_2 k_x - C_1 k_y = 0$, again leaving only the overall amplitude undetermined, since $C_1 k_x + C_2 k_y + C_3 k_z = 0$.

In either case of TE_{mnl} or TM_{mnl}, the frequency is given by

$$\omega^2 = c^2 k^2 = c^2\left(k_x^2 + k_y^2 + k_z^2\right) = c^2\pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{l}{d}\right)^2\right], \quad (5.107)$$

or

$$\omega_{mnl} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{l}{d}\right)^2}. \quad (5.108)$$