Section 5: Magnetostatics

In electrostatics, electric fields constant in time are produced by *stationary charges*. In magnetostatics magnetic fields constant in time are produced by *steady currents*.

Electric currents

The electric current in a wire is the charge per unit time passing a given point. If charge dQ passes point P in Fig.5.1 per time dt the magnitude of the current is

$$I = \frac{dQ}{dt}. \tag{5.1}$$

By definition, negative charges moving to the left count the same as positive charges moving to the right. In practice, there is a convention to assume that the electric current flows in the direction of motion of positive charges. Current is measured in coulombs-per-second, or amperes (A): 1A=1C/s.

A line charge λ traveling down a wire at speed $\mathbf{v}$ (Fig. 5.1) constitutes a current

$$\mathbf{I} = \lambda \mathbf{v}. \tag{5.2}$$

This is because a segment of length $v\Delta t$, carrying charge $\lambda v\Delta t$, passes point P in a time interval Δt.

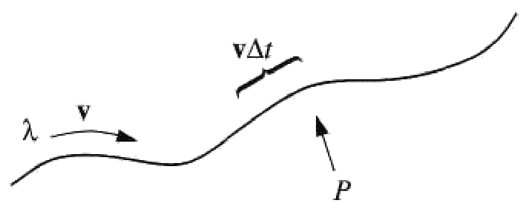


Fig. 5.1 Line current

We note that current is actually a *vector*. A neutral wire, of course, contains as many stationary positive charges as mobile negative ones. The former do not contribute to the current.

When charge flows over a surface, we describe it by the *surface current density*, **K,** defined as follows: Consider a "ribbon" of infinitesimal width $dl_\perp$, running parallel to the flow (Fig. 5.2). If the current in this ribbon is $d\mathbf{I}$, the surface current density is

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp}. \tag{5.3}$$

In words, **K** is the *current per unit width-perpendicular-to-flow*. In particular, if the mobile surface charge density is σ and its velocity is $\mathbf{v}$, then

$$\mathbf{K} = \sigma \mathbf{v}. \tag{5.4}$$

In general, **K** will vary from point to point over the surface, reflecting variations in σ and/or $\mathbf{v}$.

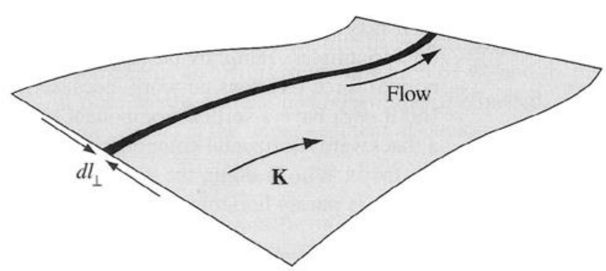


Fig. 5.2 Surface current

When the flow of charge is distributed throughout a three-dimensional region, we describe it by the *volume current density* **J** defined as follows: Consider a "tube" of infinitesimal cross section $da_\perp$, running parallel to the flow (Fig. 5.3). If the current in this tube is $d\mathbf{I}$, the volume current density is

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp} . \tag{5.5}$$

In words, **J** is the *current per unit area-perpendicular-to-flow*. If the mobile volume charge density is ρ and the velocity is **v**, then

$$\mathbf{J} = \rho \mathbf{v} . \tag{5.6}$$

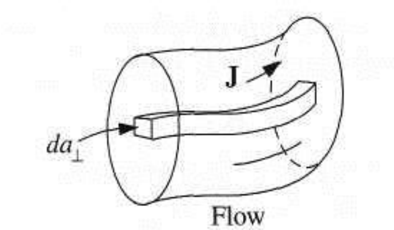


Fig. 5.3 Volume current

According to Eq.(5.5), the current crossing a surface *S* can be written as

$$I = \int_S J da_\perp = \int_S \mathbf{J} \cdot \mathbf{n} da . \tag{5.7}$$

In particular, the total charge per unit time leaving a volume *V* is

$$\oint_S \mathbf{J} \cdot \mathbf{n} da = \int_V (\nabla \cdot \mathbf{J}) d^3 r . \tag{5.8}$$

Because charge in conserved, whatever flows out through the surface must come at the expense of that remaining inside:

$$\int_V (\nabla \cdot \mathbf{J}) d^3 r = -\frac{d}{dt} \int_V \rho d^3 r = -\int_V \frac{\partial \rho}{\partial t} d^3 r . \tag{5.9}$$

The minus sign reflects the fact that an outward flow decreases the charge left in *V*. Since this applies to *any* volume, we conclude that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} . \tag{5.10}$$

This is the precise mathematical statement of local charge conservation. It is called the *continuity equation*. The continuity equation played an important role in deriving Maxwell's equations as will be discussed in electrodynamics.

Magnetostatics deals with steady currents which are characterized by no change in the net charge density anywhere in space. Consequently in magnetostatics $\partial \rho / \partial t = 0$ and therefore

$$\nabla \cdot \mathbf{J} = 0 . \tag{5.11}$$

We note here that a moving point charge does not constitute a steady current and therefore cannot be described by laws of magnetostatics.

Biot and Savart Law

It is convenient to describe magnetic phenomena in terms of a magnetic field **B**. Biot and Savart (in 1820), first, and Ampere (in 1820-1825), in much more elaborate and thorough experiments, established the basic experimental laws relating the magnetic field **B** to the electric currents and the law of force between currents.

The magnetic field is related to the current as follows. Assume that $d\mathbf{l}$ is an element of length (pointing in the direction of current flow) of a filamentary wire that carries a current **I** and **r** is the coordinate vector from the element of length to an observation point *P*, as shown in Fig. 5.4. Then the magnetic field $d\boldsymbol{B}$ at the point *P* is given by

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{d\mathbf{I}\times\mathbf{r}}{|\mathbf{r}|^3} = \frac{\mu_0}{4\pi} I \frac{d\mathbf{l}\times\mathbf{r}}{|\mathbf{r}|^3}. \tag{5.12}$$

We note that eq.(5.12) represents an inverse square law, just as is Coulomb's law of electrostatics. However, the vector character is very different. The constant $\mu_0/4\pi$ is given in SI units, where μ_0 is the permeability of free space: $\mu_0 = 4\pi\times10^7\, N/A^2$. In these units *B* itself comes out in newtons per ampere-meter or teslas (*T*): $1T = 1N/(A\cdot m)$.

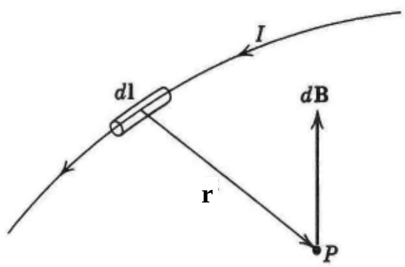


Fig. 5.4 Elemental magnetic induction $d\boldsymbol{B}$ due to current element d**l**.

In order to find the magnetic field produced by a wire carrying current *I*, we need to integrate over the wire so that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{5.13}$$

This expression is known as the *Biot and Savart law*.

Example: a straight infinite wire. By symmetry the magnetic field produced by a straight infinite wire depends only on the distance from the wire *s* and is oriented perpendicular to the wire. In Fig.5.5 the vector $\frac{d\mathbf{l}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$ points into the page and has the magnitude $dl'\cos\theta$. Since $l' = s\tan\theta$ we have $dl' = \frac{s}{\cos^2\theta}d\theta$ and also $s = |\mathbf{r}-\mathbf{r}'|\cos\theta$, so $\frac{1}{|\mathbf{r}-\mathbf{r}'|^2} = \frac{\cos^2\theta}{s^2}$. Therefore,

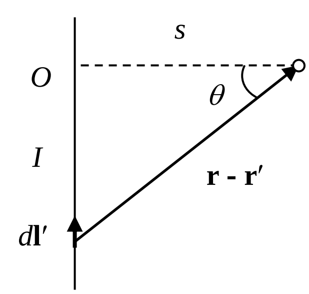


Fig. 5.5

$$B=\frac{\mu_0}{4\pi}I\int_{-\pi/2}^{\pi/2}\frac{\cos^2\theta}{s^2}\frac{s}{\cos^2\theta}\cos\theta d\theta=\frac{\mu_0 I}{4\pi s}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta=\frac{\mu_0 I}{2\pi s}\,. \tag{5.14}$$

The Biot and Savart law (5.13) can be generalized to the case of surface and volume currents.

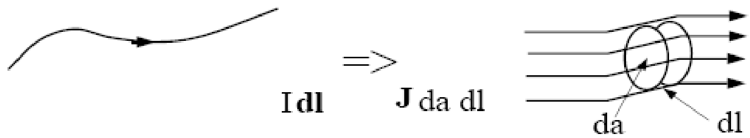


Fig. 5.6

For a volume current one has to replace $I d\mathbf{l}$ in eq. (5.13) by $\mathbf{J}(\mathbf{r})dadl$, where $\mathbf{J}(\mathbf{r})$ is the volume current density, da is the cross-sectional area of the filament, and dl is the magnitude of $d\mathbf{l}$ (see Fig.5.6). Then taking into account the fact that $dadl = d^3r$ is a volume element, we obtain

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3r'\,. \tag{5.15}$$

Similar, for surface currents we have

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{K}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}da'\,. \tag{5.16}$$

The Biot and Savart law is an analog of the Coulomb's law in electrostatics.

Ampere's experiments did not deal directly with the determination of the relation between currents and the magnetic field, but were concerned rather with the force between current-carried wires. It was experimentally found that the force experienced by a wire carried current I in the presence of a magnetic field $\mathbf{B}$ is given by

$$\mathbf{F}=\int d\mathbf{I}\times\mathbf{B}=I\int d\mathbf{l}\times\mathbf{B}\,. \tag{5.17}$$

By substituting the expression for the magnetic field (5.13) produced by another wire we find the force between two wires carrying currents I and I'

$$\mathbf{F}=\frac{\mu_0}{4\pi}II'\int\int\frac{d\mathbf{l}\times\left[d\mathbf{l}'\times(\mathbf{r}-\mathbf{r}')\right]}{|\mathbf{r}-\mathbf{r}'|^3}\,. \tag{5.18}$$

This is the original form of the *Ampere's law*. This expression can be written on a more simple form if we apply it to two current loops. Using the vector identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ we obtain

$$\mathbf{F}=\frac{\mu_0}{4\pi}II'\oint_{C'}\oint_{C}\frac{d\mathbf{l}'\left[d\mathbf{l}\cdot(\mathbf{r}-\mathbf{r}')\right]-(\mathbf{r}-\mathbf{r}')\left[d\mathbf{l}\cdot d\mathbf{l}'\right]}{|\mathbf{r}-\mathbf{r}'|^3}\,. \tag{5.19}$$

According to Newton, the force of the loop carrying current I should be equal and opposite to the force on the other loop, implying certain symmetries in the integrand above. The second term in the numerator changes sign under the interchange of $\mathbf{r}$ and $\mathbf{r}'$ (which is consistent with Newton's low), but the first term does not. It appears that the first term is identically equal to zero:

$$\oint_{C'}\oint_{C}\frac{d\mathbf{l}'\left[d\mathbf{l}\cdot(\mathbf{r}-\mathbf{r}')\right]}{|\mathbf{r}-\mathbf{r}'|^3}=\oint_{C'}d\mathbf{l}'\oint_{C}d\mathbf{l}\cdot\nabla\left(-\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)=-\oint_{C'}d\mathbf{l}'\oint_{C}d\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)=0\,. \tag{5.20}$$

Here the last step follows from the fact that we integrate over a closed path. Thus we find that the force between the two current loops may be written simply as

$$\mathbf{F} = -\frac{\mu_0}{4\pi} II' \oint_{C'} \oint_{C} \frac{d\mathbf{l} \cdot d\mathbf{l}' (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} . \tag{5.21}$$

Example: force between parallel wires

Suppose we have two parallel wires a distance d apart carrying currents I_1 and I_2 and wish to find the force per unit length acting on one of them. At a point $\mathbf{r}$ on the one carrying current I_2, there is a field $\mathbf{B}$ produced by the other current which is directed perpendicular to the plane containing the wires. This field is given by eq.(5.14) the integral over the sources in the other wire,

$$B = \frac{\mu_0 I_1}{2\pi d} . \tag{5.22}$$

According to eq. (5.17), the force is

$$F = I_2 \int B dl = \frac{\mu_0 I_1 I_2}{2\pi d} \int dl . \tag{5.23}$$

The total force is, not surprisingly, infinite, but the force per unit length is

$$f = \frac{\mu_0 I_1 I_2}{2\pi d} . \tag{5.24}$$

In case of a volume current distribution the magnetostatic force produced by a magnetic field $\mathbf{B}(\mathbf{r})$ on a object carrying a volume current density $\mathbf{J}(\mathbf{r})$ is as follows:

$$\mathbf{F} = \int d\mathbf{I} \times \mathbf{B} = \int \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) d^3 r . \tag{5.25}$$

This equation is actually a consequence of the *Lorentz force* affecting a moving charged particle. The Lorentz force is

$$\mathbf{f} = q(\mathbf{v} \times \mathbf{B}), \tag{5.26}$$

where q and $\mathbf{v}$ are a charge and velocity of the particle. According to this equation the force on the charge density $\rho(\mathbf{r})$ moving with velocity $\mathbf{v}(\mathbf{r})$ is the integral

$$\mathbf{F} = \int \mathbf{v}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) \rho(\mathbf{r}) d^3 r . \tag{5.27}$$

Taking into account that by definition $\mathbf{J}(\mathbf{r}) = \mathbf{v}(\mathbf{r})\rho(\mathbf{r})$, we arrive at eq. (5.25).

Differential Equations of Magnetostatics and Ampere's Law

To obtain differential equations for the magnetic field we rewrite Eq.(5.15) as follows:

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} d^3 r' , \tag{5.28}$$

where we have used the identity $\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$. Now using the identity

$$\nabla \times (\varphi \mathbf{a}) = (\nabla \varphi) \times \mathbf{a} + \varphi (\nabla \times \mathbf{a}), \tag{5.29}$$

where φ and $\mathbf{a}$ are arbitrary scalar and vector functions respectively, we can write

$$\nabla\times\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}=\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\times\mathbf{J}(\mathbf{r}')+\frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla\times\mathbf{J}(\mathbf{r}')=-\mathbf{J}(\mathbf{r}')\times\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\ . \tag{5.30}$$

Here we took into account that the curl is taken over the *unprimed* coordinates and therefore $\nabla\times\mathbf{J}(\mathbf{r}')=0$. This allows us to write Eq.(5.28) in the form

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\nabla\times\int\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\ . \tag{5.31}$$

It follows immediately from here that

$$\nabla\cdot\mathbf{B}=0\ . \tag{5.32}$$

This is the first differential equation of magnetostatics.

Now we calculate the curl of **B**:

$$\nabla\times\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\nabla\times\nabla\times\int\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\ . \tag{5.33}$$

Using the identity

$$\nabla\times\nabla\times\mathbf{a}=\nabla\left(\nabla\cdot\mathbf{a}\right)-\nabla^2\mathbf{a}\ , \tag{5.34}$$

which is valid for any arbitrary vector field $\mathbf{a}$, expression (5.33) can be transformed into

$$\nabla\times\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\nabla\int\mathbf{J}(\mathbf{r}')\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r'-\frac{\mu_0}{4\pi}\int\mathbf{J}(\mathbf{r}')\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r'\ . \tag{5.35}$$

If we use

$$\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}=-\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{5.36}$$

and

$$\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|}=-4\pi\delta^3\left(\mathbf{r}-\mathbf{r}'\right), \tag{5.37}$$

the integrals in (5.35) can be written:

$$\nabla\times\mathbf{B}=-\frac{\mu_0}{4\pi}\nabla\int\mathbf{J}(\mathbf{r}')\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r'+\mu_0\mathbf{J}(\mathbf{r})\ . \tag{5.38}$$

Integration by parts yields

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\frac{\mu_0}{4\pi}\nabla\left\{-\int\nabla'\cdot\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'+\int\frac{\nabla'\cdot\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\right\}. \tag{5.39}$$

The first integral in brackets vanishes because the current distribution is localized in space. This is because according to the divergence theorem it is reduced to the integral over surface bounding the volume of integration. Since we integrate over all space and since there are no currents at infinity the first integral is zero. The second integral is also equal to zero because we consider steady-state magnetic phenomena for which $\nabla\cdot\mathbf{J}=0$. Therefore we obtain

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}\ . \tag{5.40}$$

This is the second equation of magnetostatics. This is a differential form of *Ampere's law*. The integral form of the Ampere's law can be obtained by applying Stokes's theorem. Integrating the normal component of the vectors in the left- and right-hand side of Eq.(5.40) over open surface S shown in Fig.5.7 we obtain:

$$\int_S \nabla \times \mathbf{B} \cdot \mathbf{n} da = \mu_0 \int_S \mathbf{J} \cdot \mathbf{n} da \,. \tag{5.41}$$

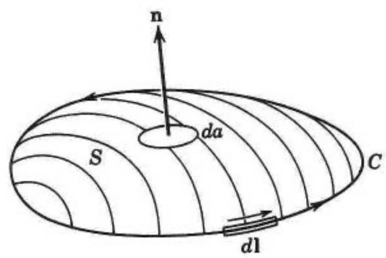


Fig. 5.7

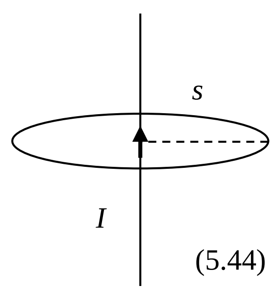
Using the Stokes's theorem it can be transformed into

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{n} da \,. \tag{5.42}$$

Since the surface integral of the current density is the total current I passing through the closed curve C, Ampere's law can be written in the form:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \,. \tag{5.43}$$

Just as Gauss's law can be used for calculation of the electric field in highly symmetric situations, so Ampere's law can be employed in analogous circumstances.



Example: a straight infinite wire. Ampere's law allows calculating the magnetic field produced by an infinite straight wire in a much more simple way as compared to the direct calculation using eq. (5.13). By symmetry the magnetic field produced depends only on the distance from the wire s and is oriented perpendicular to the wire. Using the Ampere's law we can write

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \,, \tag{5.44}$$

where C is a circle of radius s centered on the wire, and I is the current crossing the surface subtended by the circle which is exactly the current is the wire. Since $\mathbf{B}$ is constant on the circle we immediately obtain

$$2\pi s B = \mu_0 I \,, \tag{5.45}$$

or

$$B = \frac{\mu_0 I}{2\pi s} \,, \tag{5.46}$$

which is identical to the result (5.14) .

Vector Potential

The basic differential laws of magnetostatics are

$$\nabla \cdot \mathbf{B} = 0 \,, \tag{5.47}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \,. \tag{5.48}$$

According to eq. (5.47) the divergence of $\mathbf{B}$ is zero which implies that there are no sources which produce a magnetic field. There exist no magnetic analog to electric charge. According to eq. (5.48) a magnetic field curls around current. Magnetic field lines do not begin or end anywhere – to do so would require a nonzero divergence. They either form closed loops or extend out of infinity.

Now the problem is how to solve differential equations (5.47) and (5.48). If the current density is zero in the region of interest, $\nabla\times\mathbf{B}=0$ permits the expression of the magnetic field **B** as the gradient of a *magnetic scalar potential,* $\mathbf{B}=-\nabla\Phi_M$. Then (5.47) reduces to the Laplace equation for Φ_M, and all our techniques for handling electrostatic problems can be brought to bear. A large number of problems fall into this class, but we will defer discussion of them until later.

A general method of attack is to exploit equation (5.47). If $\nabla\cdot\mathbf{B}=0$ everywhere, **B** must be the curl of some vector field **A**(**r**), called the *vector potential*:

$$\mathbf{B}(\mathbf{r})=\nabla\times\mathbf{A}(\mathbf{r})\ . \tag{5.49}$$

We have, in fact, already written **B** in this form (5.31). Evidently, from (5.31), the general form of **A** is

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r+\nabla\lambda(\mathbf{r})\ . \tag{5.50}$$

The added gradient of an arbitrary scalar function λ shows that for a given magnetic induction **B**, the vector potential can be freely transformed according to

$$\mathbf{A}(\mathbf{r})\to\mathbf{A}(\mathbf{r})+\nabla\lambda(\mathbf{r})\ . \tag{5.51}$$

This transformation is called a *gauge transformation.* Such transformations on **A** are possible because (5.49) specifies only the curl of **A**. The freedom of gauge transformations allows us to make $\nabla\cdot\mathbf{A}$ have any convenient functional form we wish.

If (5.49) is substituted into the equation (5.48), we find

$$\nabla\times\nabla\times\mathbf{A}=\mu_0\mathbf{J}\ . \tag{5.52}$$

or

$$\nabla\left(\nabla\cdot\mathbf{A}\right)-\nabla^2\mathbf{A}=\mu_0\mathbf{J}\ . \tag{5.53}$$

If we now exploit the freedom implied by (5.51), we can make the convenient choice of gauge,

$$\nabla\cdot\mathbf{A}=0\ . \tag{5.54}$$

To prove that this is always possible, suppose that our original potential, $\mathbf{A}_0$, is not divergenceless. If we add to it the gradient of some scalar function λ , so that $\mathbf{A}=\mathbf{A}_0+\nabla\lambda$, the new divergence is

$$\nabla\cdot\mathbf{A}=\nabla\cdot\mathbf{A}_0+\nabla^2\lambda\ . \tag{5.55}$$

If we assume now that $\nabla\cdot\mathbf{A}=0$, then we have

$$\nabla^2\lambda=-\nabla\cdot\mathbf{A}_0\ , \tag{5.56}$$

which is represents a Poisson equation with respect to λ. The equation can be solved using standard methods that we have discussed in electrostatics. Thus we can always find such λ that gives as $\nabla\cdot\mathbf{A}=0$.

Using this gauge in eq.(5.54), we find that each rectangular component of the vector potential satisfies the Poisson equation,

$$\nabla^2\mathbf{A}=-\mu_0\mathbf{J}\ . \tag{5.57}$$

From our discussions of electrostatics it is clear that the solution for **A** in unbounded space is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' , \tag{5.58}$$

i.e. λ = constant. It is easy to see by taking directly the divergence of eq. (5.58) that indeed $\nabla \cdot \mathbf{A} = 0$.

Magnetic dipole moment

Now consider asymptotic behavior of the vector potential. Let the distribution of current be confined to a volume *V* by a surface *S*. We take $\mathbf{r}$ to outside of *V*, while $\mathbf{r}'$ is necessary inside. We examine the vector potential at large distance $r \gg r'$, starting from expression (5.58). Expanding $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ we obtain

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r} - \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}\bigg|_{\mathbf{r}'=0} \cdot \mathbf{r}' = \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} . \tag{5.59}$$

Therefore,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\int \mathbf{J}(\mathbf{r}')d^3r' + \frac{1}{r^3}\int (\mathbf{r}\cdot\mathbf{r}')\mathbf{J}(\mathbf{r}')d^3r'\right]. \tag{5.60}$$

Now we simplify this equation. First we show that

$$\int \mathbf{J}(\mathbf{r})d^3r = 0 . \tag{5.61}$$

For the localized current distribution

$$\int_V \nabla\cdot\left[x_i\mathbf{J}(\mathbf{r})\right]d^3r = \oint_S x_i\mathbf{J}(\mathbf{r})\cdot\mathbf{n}da = 0 , \tag{5.62}$$

since there are no currents crossing surface *S*. On the other hand,

$$\nabla\cdot\left[x_i\mathbf{J}(\mathbf{r})\right] = (\nabla x_i)\cdot\mathbf{J}(\mathbf{r}) + x_i\nabla\cdot\mathbf{J}(\mathbf{r}) = J_i(\mathbf{r}) , \tag{5.63}$$

where x_i is the *i* component of vector $\mathbf{r}$, and we took into account that $\nabla x_i = \hat{\mathbf{x}}_i$ and $\nabla\cdot\mathbf{J}(\mathbf{r}) = 0$. Therefore, we obtain:

$$\int \nabla\cdot\left[x_i\mathbf{J}(\mathbf{r})\right]d^3r = \int J_i(\mathbf{r})d^3r . \tag{5.64}$$

Comparing eq.(5.62) and eq. (5.63) we see that for any component *i* of the current density

$$\int J_i(\mathbf{r})d^3r = 0 , \tag{5.65}$$

which proves eq. (5.61). The first term in eq. (5.60) corresponding to the monopole term in the electrostatic expansion is therefore absent.

Second, using relation

$$\mathbf{r}\times\left[\mathbf{r}'\times\mathbf{J}(\mathbf{r}')\right] = \mathbf{r}'\left[\mathbf{r}\cdot\mathbf{J}(\mathbf{r}')\right] - (\mathbf{r}\cdot\mathbf{r}')\mathbf{J}(\mathbf{r}') , \tag{5.66}$$

we can write Eq. (5.60) as follows

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi r^3}\int (\mathbf{r}\cdot\mathbf{r}')\mathbf{J}(\mathbf{r}')d^3r' = \frac{\mu_0}{4\pi r^3}\left\{\int \mathbf{r}'\left[\mathbf{r}\cdot\mathbf{J}(\mathbf{r}')\right]d^3r' - \mathbf{r}\times\int\left[\mathbf{r}'\times\mathbf{J}(\mathbf{r}')\right]d^3r'\right\} . \tag{5.67}$$

Consider the *j* component of the first integral in eq. (5.67)

$$\int (\mathbf{r}\cdot\mathbf{J})x_j'd^3r' = \sum_i x_i\int x_j'J_id^3r' = \sum_i x_i\int x_j'\nabla'\cdot(x_i'\mathbf{J})d^3r' , \tag{5.68}$$

where we used eq.(5.63). Taking the last integral by parts we obtain

$$\int(\mathbf{r}\cdot\mathbf{J})x_j'd^3r' = \sum_i x_i\int\nabla'\cdot\left(x_j'x_i'\mathbf{J}\right)d^3r' - \sum_i x_i\int x_i'\mathbf{J}\cdot\nabla'x_j'd^3r' = -\sum_i x_i\int x_i'J_jd^3r' = -\int(\mathbf{r}\cdot\mathbf{r}')J_jd^3r', \quad (5.69)$$

Here we took into account that $\int_V \nabla'\cdot\left(x_j'x_i'\mathbf{J}\right)d^3r' = \oint_S\left(x_j'x_i'\mathbf{J}\right)\cdot\mathbf{n}da = 0$ due to no currents crossing surface *S*. Generalizing to all three components we have

$$\int\mathbf{r}'(\mathbf{r}\cdot\mathbf{J})d^3r' = -\int(\mathbf{r}\cdot\mathbf{r}')\mathbf{J}d^3r'. \quad (5.70)$$

It easy to see from Eqs.(5.67) and (5.70) that the vector potential can be written as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi r^3}\left\{-\frac{1}{2}\mathbf{r}\times\int\left[\mathbf{r}'\times\mathbf{J}(\mathbf{r}')\right]d^3r'\right\}. \quad (5.71)$$

It is customary to define *the magnetic moment density or magnetization* as

$$\mathbf{M}(\mathbf{r}) = \frac{1}{2}\left[\mathbf{r}\times\mathbf{J}(\mathbf{r})\right]. \quad (5.72)$$

and its integral as the *magnetic moment* **m**:

$$\mathbf{m} = \frac{1}{2}\int\left[\mathbf{r}\times\mathbf{J}(\mathbf{r})\right]d^3r. \quad (5.73)$$

The vector potential represents therefore the magnetic dipole vector potential:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}. \quad (5.74)$$

This is the lowest non-vanishing term in the expansion of **A** for a localized steady-state current distribution. The magnetic field **B** outside the localized source can be calculated directly by evaluating the curl of (5.74). Using the identity

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}. \quad (5.75)$$

and taking into account that **m** is constant, we obtain

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\frac{3\mathbf{r}(\mathbf{r}\cdot\mathbf{m}) - r^2\mathbf{m}}{r^5}\right]. \quad (5.76)$$

Far away from any localized current distribution the magnetic field is that of a magnetic dipole of dipole moment given by (5.73). The equation (5.76), however, is not complete for a point magnetic dipole. In this case there is an additional term that takes into account the field at the origin representing a delta-function field similar to the point electric dipole. We may find the magnitude and the direction of this singular field by a more careful analysis of what happens as $r\to 0$. To this end it is useful to write the vector potential of the dipole as follows

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla\times\left(\frac{\mathbf{m}}{r}\right), \quad (5.77)$$

where we used $\nabla\times(f\mathbf{a}) = f\nabla\times\mathbf{a} + \nabla f\times\mathbf{a}$. We can then write

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla\times\left[\nabla\times\left(\frac{\mathbf{m}}{r}\right)\right] = \frac{\mu_0}{4\pi}\left\{\nabla\left[\nabla\cdot\left(\frac{\mathbf{m}}{r}\right)\right] - \nabla^2\left(\frac{\mathbf{m}}{r}\right)\right\}. \quad (5.78)$$

The last term on the right-hand side is just $-4\pi\mathbf{m}\delta^3(\mathbf{r})$. The first one we have seen before, as it is the same as the electric field of an electric dipole and leads to eq.(5.76) when $r \neq 0$. It has also a singularity at $r = 0$. Start by integrating this term over a small sphere of radius η centered at the origin and then take the limit $\eta \to 0$:

$$\int_{r<\eta} \nabla\left[\nabla\cdot\left(\frac{\mathbf{m}}{r}\right)\right] d^3r = \int_{r=\eta} \mathbf{n}\left[\nabla\cdot\left(\frac{\mathbf{m}}{r}\right)\right] da\,. \tag{5.79}$$

Here we used the identity, valid for any scalar function $f(x)$ (as follows, e.g., from the divergence theorem: $\int_V \nabla\cdot(\mathbf{a}f)d^3r = \oint_S \mathbf{n}\cdot\mathbf{a}f da$, so that $\mathbf{a}\cdot\int_V \nabla f\, d^3r = \mathbf{a}\cdot\oint_S \mathbf{n}f da$):

$$\int_V \nabla f(\mathbf{r}) d^3r = \oint_S \mathbf{n} f(\mathbf{r}) da\,, \tag{5.80}$$

where S is the surface enclosing the domain V and $\mathbf{n}$ is the usual outward unit normal.

Continuing, we have

$$\int_{r=\eta} \mathbf{n}\left[\nabla\cdot\left(\frac{\mathbf{m}}{r}\right)\right] da = -\int_{r=\eta} \hat{\mathbf{r}}\left(\frac{\mathbf{m}\cdot\mathbf{r}}{r^3}\right) da = -\frac{4\pi}{3}\mathbf{m}\,. \tag{5.81}$$

Hence we find that $\nabla\left[\nabla\cdot\left(\frac{\mathbf{m}}{r}\right)\right]$ contains the singular piece $-\frac{4\pi}{3}\mathbf{m}$. Putting it into Eq.(5.78), we conclude that the delta-function piece of the magnetic field is $\frac{8\pi}{3}\mathbf{m}$, and hence the total field of the magnetic dipole is

$$B(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\frac{3\mathbf{r}(\mathbf{r}\cdot\mathbf{m}) - r^2\mathbf{m}}{r^5} + \frac{8\pi}{3}\mathbf{m}\delta^3(\mathbf{r})\right]. \tag{5.82}$$

Consequences of the presence of the delta-function piece are observed in an atomic hydrogen where the magnetic moment of the electron interacts with that of the nucleus, or proton. Without this interaction, all total-spin states of the atom would be degenerate. As a consequence of the interaction, the triplet (spin-one) states are raised slightly in energy relative to the singlet (spin-zero) state. The splitting is small even on the scale of atomic energies, being about 10^{-5} eV. The delta-function part of the field also plays an important role in the scattering of neutrons from magnetic materials.

If the current is confined to a plane, but otherwise arbitrary, loop, the magnetic moment can be expressed in a simple form. If the current I flows in a closed circuit whose line element is d**l**, (5.72) becomes

$$\mathbf{m} = \frac{I}{2}\oint \mathbf{r}\times d\mathbf{l}\,. \tag{5.83}$$

For a plane loop such as that in Fig. 5.8, the magnetic moment is perpendicular to the plane of the loop. Since $\tfrac{1}{2}|\mathbf{r}\times d\mathbf{l}| = da$, where d$a$ is the triangular element of the area defined by the two ends of d**l** and the origin, the loop integral gives the total area of the loop. Hence the magnetic moment has magnitude,

$$|\mathbf{m}| = I\times(\text{Area})\,,$$

regardless of the shape of the circuit.

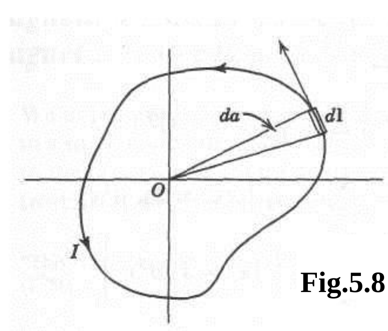


Fig.5.8

Forces on a Localized Current Distribution

If a localized current distribution is placed in an external magnetic field $\mathbf{B}(\mathbf{r})$, it experiences a force according to Ampere's law. The general expression for the force is given by

$$\mathbf{F} = \int \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) d^3 r . \tag{5.85}$$

If the external magnetic field varies slowly over the region of current, a Taylor series expansion can be used to find the dominant term s in the force. We expand the applied field around some suitably chosen origin (within the current distribution) so that

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(0) + (\mathbf{r} \cdot \nabla') \mathbf{B}(\mathbf{r}')\big|_{\mathbf{r}'=0} + \ldots . \tag{5.86}$$

The force that the field exerts on a localized current distribution located around the origin is then expanded as follows:

$$\mathbf{F} = -\mathbf{B}(0) \times \int \mathbf{J}(\mathbf{r}) d^3 r + \int \mathbf{J}(\mathbf{r}) \times (\mathbf{r} \cdot \nabla') \mathbf{B}(\mathbf{r}')\big|_{\mathbf{r}'=0} d^3 r + \ldots . \tag{5.87}$$

Now, the first integral in the last line vanishes for a localized steady state current distribution (there can't be any net flow of charge in any direction), and we can manipulate the integrand in the final integral as follows:

$$(\mathbf{r} \cdot \nabla') \mathbf{B}(\mathbf{r}') = \nabla' (\mathbf{r} \cdot \mathbf{B}(\mathbf{r}')) - \mathbf{r} \times [\nabla' \times \mathbf{B}(\mathbf{r}')]. \tag{5.88}$$

and we may suppose that $\mathbf{B}$ is due entirely to external sources so that $\nabla' \times \mathbf{B}(\mathbf{r}') = 0$ around the origin. Thus, using identity $\nabla \times (f\mathbf{a}) = f(\nabla \times \mathbf{a}) + \nabla f \times \mathbf{a}$, we find that the force is

$$\mathbf{F} = \int \mathbf{J}(\mathbf{r}) \times \nabla' (\mathbf{r} \cdot \mathbf{B}(\mathbf{r}'))\big|_{\mathbf{r}'=0} d^3 r = -\int \nabla' \times \Big[(\mathbf{r} \cdot \mathbf{B}(\mathbf{r}'))\big|_{\mathbf{r}'=0} \mathbf{J}(\mathbf{r}) \Big] d^3 r = -\nabla' \times \int \Big[(\mathbf{r} \cdot \mathbf{B}(\mathbf{r}'))\big|_{\mathbf{r}'=0} \mathbf{J}(\mathbf{r}) \Big] d^3 r . \tag{5.89}$$

Now, we can write the last integral as

$$-\int (\mathbf{r} \cdot \mathbf{B}(\mathbf{r}')) \mathbf{J}(\mathbf{r}) d^3 r = \int \mathbf{B}(\mathbf{r}') \times (\mathbf{r} \times \mathbf{J}(\mathbf{r})) d^3 r - \int \mathbf{r} (\mathbf{B}(\mathbf{r}') \cdot \mathbf{J}(\mathbf{r})) d^3 r . \tag{5.90}$$

Further, we show that

$$\int \mathbf{r} (\mathbf{B}(\mathbf{r}') \cdot \mathbf{J}(\mathbf{r})) d^3 r = -\int (\mathbf{r} \cdot \mathbf{B}(\mathbf{r}')) \mathbf{J}(\mathbf{r}) d^3 r . \tag{5.91}$$

This can be demonstrated by considering the i component of this integral and integrating by parts:

$$\begin{aligned} &-\int (\mathbf{r} \cdot \mathbf{B}(\mathbf{r}')) J_i(\mathbf{r}) d^3 r = -\int (\mathbf{r} \cdot \mathbf{B}(\mathbf{r}')) \nabla \cdot (x_i \mathbf{J}(\mathbf{r})) d^3 r = \\ &= \int \nabla (\mathbf{r} \cdot \mathbf{B}(\mathbf{r}')) \cdot (x_i \mathbf{J}(\mathbf{r})) d^3 r = \int \mathbf{B}(\mathbf{r}') \cdot (x_i \mathbf{J}(\mathbf{r})) d^3 r = \int x_i (\mathbf{B}(\mathbf{r}') \cdot \mathbf{J}(\mathbf{r})) d^3 r. \end{aligned} \tag{5.92}$$

Here we took into account that $\nabla (\mathbf{r} \cdot \mathbf{B}(\mathbf{r}')) = \sum_{ij} \frac{\partial}{\partial x_i} (x_j B_j) \hat{\mathbf{x}}_i = \sum_{ij} \delta_{ij} B_j \hat{\mathbf{x}}_i = \mathbf{B}$. Hence from eqs. (5.90), and (5.91), we find

$$-\int (\mathbf{r} \cdot \mathbf{B}(\mathbf{r}')) \mathbf{J}(\mathbf{r}) d^3 r = \frac{1}{2} \int \mathbf{B}(\mathbf{r}') \times (\mathbf{r} \times \mathbf{J}(\mathbf{r})) d^3 r = \mathbf{B}(\mathbf{r}') \times \mathbf{m} . \tag{5.93}$$

Now using eq. (5.93), from eq. (5.89) we obtain

$$\begin{aligned} \mathbf{F} &= \{ \nabla' \times \mathbf{B}(\mathbf{r}') \times \mathbf{m} \}_{\mathbf{r}'=0} = (\mathbf{m} \cdot \nabla') \mathbf{B}(\mathbf{r}')\big|_{\mathbf{r}'=0} - \mathbf{m} (\nabla' \cdot \mathbf{B}(\mathbf{r}'))\big|_{\mathbf{r}'=0} = \\ &= \nabla' (\mathbf{m} \cdot \mathbf{B}(\mathbf{r}'))\big|_{\mathbf{r}'=0} - \mathbf{m} \times (\nabla' \times \mathbf{B}(\mathbf{r}'))\big|_{\mathbf{r}'=0} = \nabla' (\mathbf{m} \cdot \mathbf{B}(\mathbf{r}'))\big|_{\mathbf{r}'=0} . \end{aligned} \tag{5.94}$$

Along the way in this derivation, we have made use of the facts that the divergence and curl of **B** are zero in the region near the origin. The final result has the form of the gradient of a scalar function,

$$\mathbf{F} = \nabla\left(\mathbf{m}\cdot\mathbf{B}\right), \tag{5.95}$$

where the gradient is to be evaluated at the center of the current distribution. Notice in particular that there is no force if the applied magnetic induction is uniform. More generally, the force is in the direction of the gradient of the component of **B** in the direction of **m**.

The potential energy of a permanent magnetic moment (dipole) in an external magnetic field can be obtained from the expression for the force (5.95). If we interpret the force as the negative gradient of the potential energy *U*, we find

$$U = -\mathbf{m}\cdot\mathbf{B}. \tag{5.96}$$

This is well-known result, which shows that the dipole tends to orient itself parallel to the field in the position of lowest potential energy.