

## Section 4: Electrostatics of Dielectrics

### Dielectrics and Polarizability

There are two large classes of substances: conductors and insulators (or dielectrics). In contrast to metals where charges are free to move throughout the material, in dielectrics all the charges are attached to specific atoms and molecules. These charges are known as *bound* charges. These charges are able, however, to be displaced within an atom or a molecule. Such microscopic displacements are not as dramatic as the rearrangement of charge in a conductor, but their cumulative effects account for the characteristic behavior of dielectric materials.

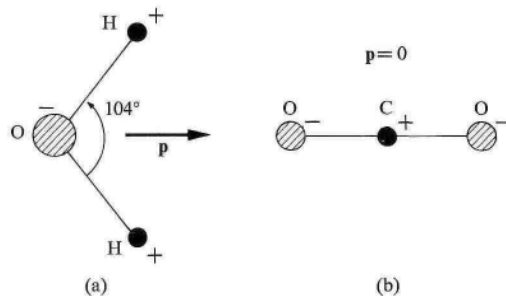
When an external electric field is applied to a dielectric material this material becomes *polarized*, which means that it acquires a *dipole moment*. This property of dielectrics is known as *polarizability*. Basically, polarizability is a consequence of the fact that molecules, which are the building blocks of all substances, are composed of both positive charges (nuclei) and negative charges (electrons). When an electric field acts on a molecule, the positive charges are displaced along the field, while the negative charges are displaced in a direction opposite to that of the field. The effect is therefore to pull the opposite charges apart, i.e., to polarize the molecule.

It is convenient to define the *polarizability*  $\alpha$  of an atom in terms of the local electric field at the atom:

$$\mathbf{p} = \alpha \mathbf{E}_{loc} . \quad (4.1)$$

where  $\mathbf{p}$  is the dipole moment. For a non-spherical atom  $\alpha$  will be a tensor.

There are different types of polarization processes, depending on the structure of the molecules which constitute the solid. If the molecule has a *permanent moment*, i.e., a moment is present even in the absence of an electric field, we speak of a *dipolar* molecule, and a *dipolar* substance.

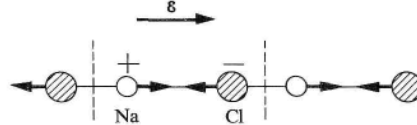


**Fig.4.1** (a) The water molecule (b) CO<sub>2</sub> molecule.

An example of a dipolar molecule is the H<sub>2</sub>O molecule in Fig.4.1a. The dipole moments of the two OH bonds add vectorially to give a nonvanishing net dipole moment. Some molecules are nondipolar, possessing no permanent moments; a common example is the CO<sub>2</sub> molecule in Fig.4.1b. The moments of the two CO bonds cancel each other because of the rectilinear shape of the molecule, resulting in a zero net dipole moment in the absence of electric field.

Despite the fact that the individual molecules in a dipolar substance have permanent moments, the net polarization vanishes in the absence of an external field because the molecular moments are randomly oriented, resulting in a complete cancellation of the polarization. When a field is applied to the substance, however, the molecular dipoles tend to align with the field. The reason is that the energy of a dipole  $\mathbf{p}$  in a local external field  $\mathbf{E}_{loc}$  is  $U = -\mathbf{p} \cdot \mathbf{E}_{loc}$ . It has a minimum when the dipole is parallel to the field. This results in a net non-vanishing dipole moment of the material. This mechanism for polarizability is called *dipolar polarizability*.

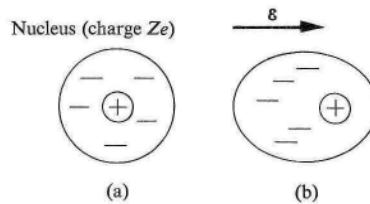
If the molecule contains ionic bonds, then the field tends to stretch the lengths of these bonds. This occurs in NaCl, for instance, because the field tends to displace the positive ion  $\text{Na}^+$  to the right (see Fig.4.2), and the negative ion  $\text{Cl}^-$  to the left, resulting in a stretching in the length of the bond. The effect of this change in length is to produce a net dipole moment in the unit cell where previously there was none. Since the polarization here is due to the relative displacements of oppositely charged ions, we speak of *ionic polarizability*.



**Fig.4.2** Ionic polarization in NaCl. The field displaces  $\text{Na}^+$  and  $\text{Cl}^-$  ions in opposite directions, changing the bond length.

Ionic polarizability exists whenever the substance is either ionic, as in NaCl, or dipolar, as in  $\text{H}_2\text{O}$ , because in each of these classes there are ionic bonds present. But in substances in which such bonds are missing - such as Si and Ge - ionic polarizability is absent.

The third type of polarizability arises because the individual ions or atoms in a molecule are themselves polarized by the field. In the case of NaCl, each of the  $\text{Na}^+$  and  $\text{Cl}^-$  ions are polarized. Thus the  $\text{Na}^+$  ion is polarized because the electrons in its various shells are displaced to the left relative to the nucleus, as shown in Fig.4.3. We are clearly speaking here of *electronic polarizability*.



**Fig. 4.3** Electronic polarization: (a) Unpolarized atom, (b) Atom polarized as a result of the field.

Electronic polarizability arises even in the case of a neutral atom, again because of the relative displacement of the orbital electrons.

In general, therefore, the total polarizability is given by

$$\alpha = \alpha_e + \alpha_i + \alpha_d, \tag{2}$$

which is the sum of the electronic, ionic, and dipolar polarizabilities, respectively. The electronic contribution is present in any type of substance, but the presence of the other two terms depends on the material under consideration.

The relative magnitudes of the various contributions in (2) are such that in nondipolar, ionic substances the electronic part is often of the same order as the ionic. In dipolar substances, however, the greatest contribution comes from the dipolar part. This is the case for water, for example.

### Polarization

If electric field is applied to a medium made up of large number of atoms or molecules, the charges bound in each molecule will respond to applied field which will results in the redistribution of charges leading to a *polarization* of the medium. The electric polarization  $\mathbf{P}(\mathbf{r}')$  is defined as the dipole moment per unit volume. The polarization is a macroscopic quantity because it involves averaging of the dipole moments over a volume which contains many dipoles. We assume that the response of the system to an applied field is linear. This excludes ferroelectricity from discussion, but otherwise is no real restriction provided the field

strengths do not become extremely large. As a further simplification we suppose that the *medium is isotropic*. Then the induced polarization  $\mathbf{P}$  is parallel to  $\mathbf{E}$  with a coefficient of proportionality that is independent of direction:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}. \quad (4.3)$$

The constant  $\chi_e$  is the *electric susceptibility* of the medium.

An important point to note that the electric field which enters eq. (4.3) is the a macroscopic electric field which is different from a local electric field entering eq. (4.1). The macroscopic field is the average over volume with a size large compared to an atomic size.

Now we look at the medium from a macroscopic point of view assuming that the medium contains free charges characterized by the charge density  $\rho$  and bound charges characterized by polarization  $\mathbf{P}$ . We can build up the potential and the field by linear superposition of the contributions from each macroscopically small volume element  $\delta V$  at the variable point  $\mathbf{r}'$ . The free charge contained in volume  $\delta V$  is  $\rho(\mathbf{r}') \delta V$  and the dipole moment of  $\delta V$  is  $\mathbf{P}(\mathbf{r}') \delta V$ . If there are no higher macroscopic multipole moment densities, the contribution to the potential  $\delta\Phi(\mathbf{r}, \mathbf{r}')$  caused by the configuration of moments in  $\delta V$  is given without approximation by

$$\delta\Phi(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0} \left[ \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \delta V + \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \delta V \right]. \quad (4.4)$$

provided  $\mathbf{r}$  is outside  $\delta V$ . The first term is the contribution from free charges and the second term is due to a volume distribution of dipoles. We now treat  $\delta V$  as (macroscopically) infinitesimal, put it equal to  $d^3 r'$ , and integrate over the volume of the dielectric to obtain the potential

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \left[ \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right] d^3 r'. \quad (4.5)$$

To simplify this equation we use the identity

$$\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = -\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|}, \quad (4.6)$$

where  $\nabla'$  implies differentiation with respect to  $\mathbf{r}'$ . This allows us to rewrite Eq. (4.5) as follows:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \left[ \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \mathbf{P}(\mathbf{r}') \cdot \nabla' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] d^3 r'. \quad (4.7)$$

We can take this integral by parts. Taking into account

$$\nabla' \cdot \left\{ \frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right\} = \mathbf{P}(\mathbf{r}') \cdot \nabla' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) + \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbf{P}(\mathbf{r}'), \quad (4.8)$$

an integration by parts transforms the potential into

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \left\{ \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbf{P}(\mathbf{r}') + \nabla' \cdot \left[ \frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right] \right\} d^3 r'. \quad (4.9)$$

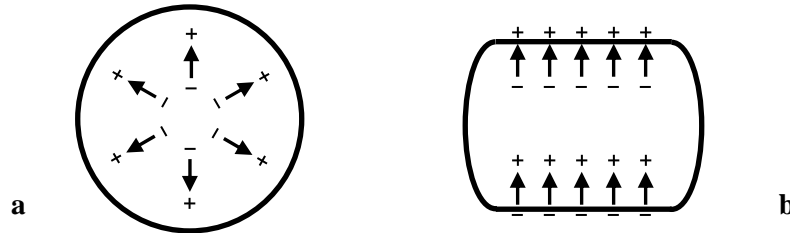
We can now use the divergence theorem to transform the third term in eq.(4.9) to the integral over surface of the dielectric, which results in

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{d^3r'}{|\mathbf{r}-\mathbf{r}'|} [\rho(\mathbf{r}') - \nabla' \cdot \mathbf{P}(\mathbf{r}')] + \frac{1}{4\pi\epsilon_0} \oint_S \frac{\mathbf{P}(\mathbf{r}') \cdot \mathbf{n}}{|\mathbf{r}-\mathbf{r}'|} da. \quad (4.10)$$

As follows from this expression, the polarization of the medium produces an effective charge which can be interpreted as a macroscopic *bound* charge or *polarization* charge. There are two contributions to the bound charge – bulk and surface. The *volume charge density* is given by

$$\rho_p(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r}). \quad (4.11)$$

The presence of the divergence of  $\mathbf{P}$  in the effective charge density can be understood qualitatively. If the polarization is nonuniform there can be a net increase or decrease of charge within any small volume. For example in Fig.4.4a, at the center of this region, where the tails of the dipoles are concentrated, there is an excess of negative charge.



**Fig. 4.4** Origin of polarization-charge density. (a) bulk charge density due to the divergence of polarization; (b) surface charge density due to uncompensated charges of the surface.

The *surface charge density* is

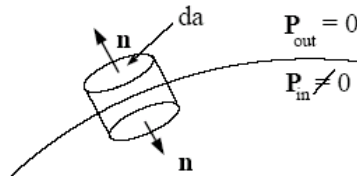
$$\sigma_p(\mathbf{r}) = \mathbf{P}(\mathbf{r}) \cdot \mathbf{n}. \quad (4.12)$$

This contribution is present even for the uniform polarization within a finite volume. In this case the average polarization charge inside the dielectric is zero, because if we take a macroscopic volume, it will contain equal amount of positive and negative charges and the net charge will be zero. On the other hand if we consider a volume including a boundary perpendicular to the direction of polarization, there is a net positive (negative) charge on the surface which is not compensated by charges inside the dielectric, as is seen in Fig.4b. Therefore, the polarization charge appears on the surface on the dielectric.

In deriving Eq.(4.10) we can integrate over all the space. In this case the surface integral (the third term in this equation) vanishes due to the assumption of the finite volume of the dielectric. The expression for the potential then becomes

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{all\ space} \frac{d^3r'}{|\mathbf{r}-\mathbf{r}'|} [\rho(\mathbf{r}') - \nabla' \cdot \mathbf{P}(\mathbf{r}')]. \quad (4.13)$$

In this case, the surface polarization charge (4.12) is implicitly included in  $-\nabla \cdot \mathbf{P}(\mathbf{r})$  due to the abrupt change of the polarization at the surface. This can be seen from the following consideration.



**Fig.4.5**

Assume that polarization  $\mathbf{P}(\mathbf{r})$  has discontinuity at the surface as is shown in Fig.4.5. Consider a small pill box enclosing a small section of the volume and surface of the polarized material. From the divergence theorem we have

$$\int_V \nabla \cdot \mathbf{P}(\mathbf{r}) d^3r = \int_S \mathbf{P}(\mathbf{r}) \cdot \mathbf{n} da . \quad (4.14)$$

If the region is small enough this results in

$$(\mathbf{P}_{out} \cdot \mathbf{n} - \mathbf{P}_{in} \cdot \mathbf{n}) da = \nabla \cdot \mathbf{P} d^3r . \quad (4.15)$$

Taking into account that  $\mathbf{P}_{out} = 0$  and  $\mathbf{P}_{in} = \mathbf{P}$  we have

$$\mathbf{P} \cdot \mathbf{n} da = -\nabla \cdot \mathbf{P} d^3r , \quad (4.16)$$

Thus  $-\nabla \cdot \mathbf{P}$  must have a delta-function at the surface, and we still have the surface polarization charge so that on the surface

$$\sigma_p da = \rho_p d^3r . \quad (4.17)$$

Thus, the polarization charge can always be represented by

$$\rho_p = -\nabla \cdot \mathbf{P} . \quad (4.18)$$

It is important to note that the total polarization charge is always equal to zero. This is the consequence of the charge conservation – by inducing an electric polarization in a material we do not change the total charge. Mathematically this fact can be easily seen from eq.(4.18) – the integration of the polarization charge over any closed surface which enclosed the volume of a polarized material gives zero according to the divergence theorem.

We can, therefore, make a general statement that the presence of the polarization produces an additional polarization charge so that the total charge density becomes

$$\rho_{total} = \rho_{free} + \rho_p = \rho - \nabla \cdot \mathbf{P} . \quad (4.19)$$

We can therefore, rewrite the expression for the divergence of  $\mathbf{E}$  as follows:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} [\rho - \nabla \cdot \mathbf{P}] . \quad (4.20)$$

It is convenient to define the *electric displacement*  $\mathbf{D}$ ,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} , \quad (4.21)$$

Because this field is generated is generated by free charges only. Using the electric displacement the Gauss's law takes the form

$$\nabla \cdot \mathbf{D} = \rho . \quad (4.22)$$

In the integral form it reads as follows:

$$\oint_S \mathbf{D} \cdot \mathbf{n} da = \int_V \rho(\mathbf{r}) d^3r . \quad (4.23)$$

This is particularly useful way to represent Gauss's law because it makes reference *only on free charges*.

Connecting  $\mathbf{D}$  and  $\mathbf{E}$  is necessary before a solution for the electrostatic potential or fields can be obtained. For a linear response of the system (4.3) the displacement  $\mathbf{D}$  is proportional to  $\mathbf{E}$ ,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon \mathbf{E} , \quad (4.24)$$

where

$$\epsilon = \epsilon_0 (1 + \chi_e) . \quad (4.25)$$

is the electric permittivity;  $\epsilon_r = \epsilon / \epsilon_0 = 1 + \chi_e$  is called the *dielectric constant* or relative electric permittivity.

If the dielectric is not only isotropic, but also uniform, then  $\epsilon$  is independent of position. The Gauss's law (4.22) can then be written

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}. \quad (4.26)$$

In this case all problems *in that medium* are reduced to those with no electric polarization, except that the electric fields produced by given charges are reduced by a factor  $\epsilon/\epsilon_0$ . The reduction can be understood in terms of a polarization of the atoms that produce fields in opposition to that of the given charge. One immediate consequence is that the capacitance of a capacitor is increased by a factor of  $\epsilon/\epsilon_0$  if the empty space between the electrodes is filled with a dielectric with dielectric constant  $\epsilon/\epsilon_0$ .

Now we consider a couple of simple examples.

1. A slab with a uniform polarization pointing perpendicular to the surface

Assume that we have a slab of a dielectric which has a uniform polarization pointing in the  $z$  direction (Fig.4.6). In this case we have non-zero surface polarization charges,  $\sigma_p = +\mathbf{P}$  on the top surface and  $\sigma_p = -\mathbf{P}$  on the bottom surface, resulting in the electric field  $\mathbf{E} = -\mathbf{P}/\epsilon_0$  in the slab and  $\mathbf{E} = 0$  outside the slab. The electric displacement  $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = 0$  is zero everywhere.

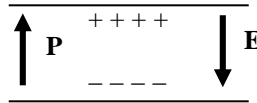


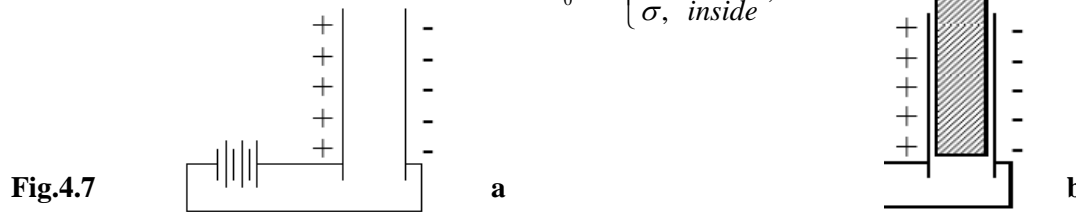
Fig.4.6

If the polarization is uniform and parallel to the surfaces then the electric field  $\mathbf{E}$  is zero, everywhere is space and thus  $\mathbf{D} = \mathbf{P}$  inside the slab and  $\mathbf{D} = 0$  outside.

2. Capacitor with a dielectric material inside

Another simple example is the parallel plate capacitor (Fig.4.7a). Assume that a battery voltage is applied to the plates of the capacitor so that the plates acquire a surface charge  $\sigma$ . In the absence of dielectric material the polarization is zero in all the space and therefore

$$\mathbf{D} = \epsilon_0 \mathbf{E} = \begin{cases} 0, & \text{outside} \\ \sigma, & \text{inside} \end{cases}, \quad (4.27)$$



Now we remove the battery so that the surface charge is fixed and slide a dielectric slab between the plates (Fig.4.7b). The dielectric material obtains a uniform polarization, giving rise to surface polarization charges. However,  $\mathbf{D}$  responds to only free charges, thus it is unchanged by the introduction of the dielectric slab.  $\mathbf{E}$  responds to all charges, so it changes. Since  $\mathbf{E} = (\mathbf{D} - \mathbf{P})/\epsilon_0$ , and  $\mathbf{P}$  is parallel to  $\mathbf{E}$ , we see that  $\mathbf{E}$  decreases in magnitude. The ratio of the electric field between the plates before and after the dielectric was introduced is

$$\frac{\mathbf{E}_{dielectric}}{\mathbf{E}_{vacuum}} = \frac{\mathbf{E}_{dielectric}}{\mathbf{D}/\epsilon_0} = \frac{\mathbf{E}_{dielectric}}{\epsilon \mathbf{E}_{dielectric}/\epsilon_0} = \frac{1}{\epsilon_r}, \quad (4.28)$$

where  $\epsilon_r$  is the dielectric constant.

### 3. A sphere with a radial polarization distribution

Another simple example is a sphere with the radial distribution of polarization (Fig.4.8). The magnitude of the polarization is constant and only the direction changes. Thus, the polarization is

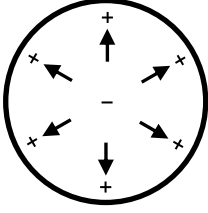


Fig.4.8

$$\mathbf{P} = \begin{cases} 0, & \text{outside} \\ P\hat{\mathbf{r}}, & \text{inside} \end{cases} \quad (4.29)$$

This polarization obviously produces a surface polarization charge on the surface of the sphere which is equal to  $\sigma_p = \mathbf{P} \cdot \mathbf{n} = P$ . The volume polarization charge can be found using eq.(4.18) which leads to

$$\rho_p = -\nabla \cdot \mathbf{P} = -P\nabla \cdot \hat{\mathbf{r}} = -P \frac{1}{r^2} \frac{\partial}{\partial r} (r^2) = -\frac{2P}{r}. \quad (4.30)$$

We see that the volume polarization charge is distributed over all the sphere and diverges at the center of the sphere. The total polarization charge is zero. Indeed, the total volume polarization charge is

$$\int \rho_p d^3r = \int_0^R \left( -\frac{2P}{r} \right) 4\pi r^2 dr = -4\pi R^2 P, \quad (4.31)$$

the total surface polarization charge is  $4\pi R^2 P$ .

### 4. Polarization of a point charge.

Consider a positive point charge  $q$  placed at the origin of an infinite dielectric medium of electric permittivity  $\epsilon$ . We need to find polarization and polarization charges. Since the system has a spherical symmetry, we can use a Gauss's law (4.22) to find the electric displacement  $\mathbf{D}$ :

$$\oint_S \mathbf{D} \cdot \mathbf{n} da = q, \quad (4.32)$$

where the integration is performed over a sphere of radius  $r$  centered at the origin. This leads to

$$\mathbf{D} = \frac{q}{4\pi r^2} \hat{\mathbf{r}}, \quad (4.33)$$

Consequently the electric field is

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{q}{4\pi\epsilon r^2} \hat{\mathbf{r}}. \quad (4.34)$$

According to eq. (4.21), the polarization is

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = \frac{q}{4\pi r^2} \hat{\mathbf{r}} - \frac{\epsilon_0 q}{4\pi\epsilon r^2} \hat{\mathbf{r}} = \frac{q}{4\pi} \left( 1 - \frac{\epsilon_0}{\epsilon} \right) \frac{\hat{\mathbf{r}}}{r^2} = \frac{q}{4\pi} \left( \frac{\epsilon - \epsilon_0}{\epsilon} \right) \frac{\hat{\mathbf{r}}}{r^2}, \quad (4.35)$$

The polarization charge is

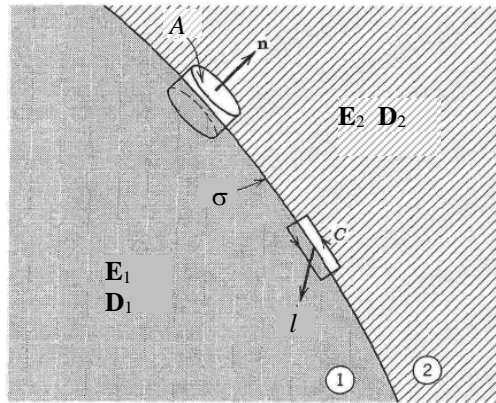
$$\rho_p = -\nabla \cdot \mathbf{P} = -\frac{q}{4\pi} \left( \frac{\epsilon - \epsilon_0}{\epsilon} \right) \nabla \cdot \left( \frac{\hat{\mathbf{r}}}{r^2} \right) = -q \left( \frac{\epsilon - \epsilon_0}{\epsilon} \right) \delta^3(\mathbf{r}). \quad (4.36)$$

We have therefore a polarization at the origin which has an opposite sign to the free charge  $q$ . The sum of

the free and polarization charge is  $\frac{\epsilon_0}{\epsilon}q = \frac{q}{\epsilon_r}$ , which gives a correct electric field (4.34) if we use this charge in Coulomb's law. We note that the total polarization charge is not zero because the system considered is infinite and hence the positive polarization charge are located at infinity.

### Boundary conditions

For solving electrostatics problems one needs to know boundary conditions for the electric field.



**Fig. 4.9** Schematic diagram of the boundary surface between different media.

Consider a boundary between different media, as is shown in Fig.4.9. The boundary region is assumed to carry idealized surface charge  $\sigma$ . Consider a small pillbox, half in one medium and half in the other, with the normal it to its top pointing from medium 1 into medium 2. According to the Gauss's law

$$\oint_S \mathbf{D} \cdot \mathbf{n} da = \sigma A, \quad (4.37)$$

where the integral is taken over the surface of the pillbox and  $A$  is the area of the pillbox lid. In the limit of zero thickness the sides of the pillbox contribute nothing to the flux. The contribution from the top and bottom surfaces to the integral gives  $A(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n}$ , resulting in

$$D_2^\perp - D_1^\perp = \sigma, \quad (4.38)$$

where  $D^\perp$  is the component of the electrical displacement perpendicular to the surface. Eq. (4.38) tells us that there is a discontinuity of the  $D^\perp$  at the interface which is determined by the surface charge.

Now we consider a rectangular contour  $C$  such that it is partly in one medium and partly in the other and is oriented with its plane perpendicular to the surface. Since the curl of electric field is zero we have

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0. \quad (4.39)$$

For the rectangular contour  $C$  of infinitesimal height this integral is equal to  $(E_2^\parallel - E_1^\parallel)l$ , where  $E^\parallel$  is the component of electric field parallel to the surface. This implies that

$$\mathbf{E}_2^\parallel = \mathbf{E}_1^\parallel, \quad (4.40)$$

i.e. the tangential component of electric field is always continuous.

The electrostatic potential is continuous across the boundary. Indeed, if we consider two points, one above the surface,  $\mathbf{a}$ , and the other below the surface,  $\mathbf{b}$ , then



$$\Phi(\mathbf{b}) - \Phi(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}. \quad (4.41)$$

As the path length shrinks to zero, the integral does too.

### Boundary value problems with dielectrics

Since the presence of dielectrics equations for the electric field are  $\nabla \times \mathbf{E} = 0$  and  $\nabla \cdot \mathbf{E} = \rho/\epsilon$ , the electrostatic potential defined by  $\mathbf{E} = -\nabla\Phi$  obeys the Poisson equation

$$\nabla^2\Phi = -\rho/\epsilon. \quad (4.42)$$

Therefore, all the methods of solving boundary value problems in electrostatics discussed in preceding sections can readily be extended to dielectric materials. Below we consider a few examples.

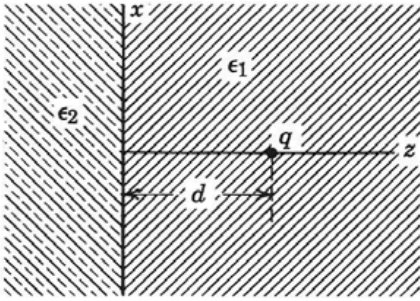


Fig.4.10a

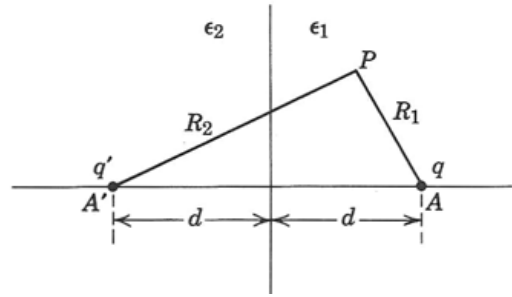


Fig.4.10b

**Example 1:** To illustrate the method of images for dielectrics we consider a point charge  $q$  embedded in a semi-infinite dielectric, with dielectric constant  $\epsilon_1$ , a distance  $d$  away from a plane interface that separates the first medium from another semi-infinite dielectric  $\epsilon_2$ . The surface may be taken as the plane  $z = 0$ , as shown in Fig.4.10a. We must find the appropriate solution to the equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_1}, \quad z > 0. \quad (4.43)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_2}, \quad z < 0. \quad (4.44)$$

and

$$\nabla \times \mathbf{E} = 0, \quad (4.45)$$

subject to the boundary conditions (4.38) and (4.40) at  $z = 0$ :

$$D_2^\perp = \epsilon_2 E_{2z} = D_1^\perp = \epsilon_1 E_{1z} \quad (4.46)$$

$$\begin{aligned} E_{2x} &= E_{1x} \\ E_{2y} &= E_{1y} \end{aligned} \quad (4.47)$$

where we took in to account that  $\sigma = 0$ . Since  $\nabla \times \mathbf{E} = 0$  everywhere,  $\mathbf{E}$  can be obtained from the potential  $\mathbf{E} = -\nabla\Phi$ . We attempt to use the image method by locating an image charge  $q'$  at the symmetrical position  $A'$  shown in Fig.4.10b. Then for  $z > 0$  the potential at a point  $P$  described by cylindrical coordinates  $(s, \phi, z)$  will be

$$\Phi = \frac{1}{4\pi\epsilon_1} \left( \frac{q}{R_1} + \frac{q'}{R_2} \right), \quad z > 0. \quad (4.48)$$

where  $R_1 = \sqrt{s^2 + (d - z)^2}$  and  $R_2 = \sqrt{s^2 + (d + z)^2}$ . So far the procedure is completely analogous to the problem with a conducting material in place of the dielectric  $\epsilon_2$  for  $z < 0$ . But we now must specify the potential for  $z < 0$ . Since there are no charges in the region  $z < 0$ , it must be a solution of the Laplace equation without singularities in that region. Clearly the simplest assumption is that for  $z < 0$  the potential is equivalent to that of a charge  $q''$  at the position  $A$  of the actual charge  $q$ :

$$\Phi = \frac{1}{4\pi\epsilon_2} \frac{q''}{R_1}, \quad z < 0. \quad (4.49)$$

Now we try to pick the image charges in such a way that the boundary conditions (4.46) and (4.47) are satisfied. These conditions involve the following derivatives:

$$\frac{\partial}{\partial z} \left( \frac{1}{R_1} \right)_{z=0} = -\frac{\partial}{\partial z} \left( \frac{1}{R_2} \right)_{z=0} = \frac{d}{(s^2 + d^2)^{3/2}}. \quad (4.50)$$

and

$$\frac{\partial}{\partial s} \left( \frac{1}{R_1} \right)_{z=0} = \frac{\partial}{\partial s} \left( \frac{1}{R_2} \right)_{z=0} = \frac{-s}{(s^2 + d^2)^{3/2}}. \quad (4.51)$$

Using these equations we find that the condition for the continuous normal component of  $\mathbf{D}$  (4.46), which is determined by the  $z$ -component of electric field  $E_z = -\partial\Phi/\partial z$ , leads to

$$q - q' = q'', \quad (4.52)$$

and that the condition of continuous tangential component of  $\mathbf{E}$  (4.47), which is given by  $E_s = -\frac{\partial\Phi}{\partial s}$ , results in

$$\frac{(q + q')}{\epsilon_1} = \frac{q''}{\epsilon_2}. \quad (4.53)$$

These equations can be solved to yield the image charges  $q'$  and  $q''$ :

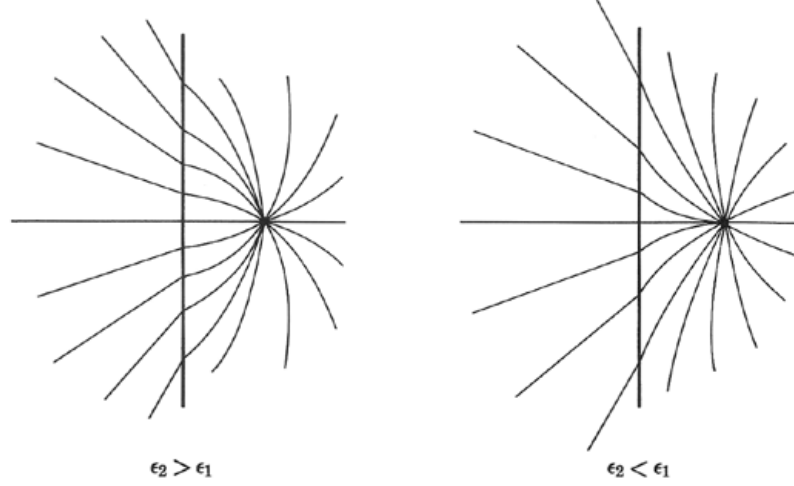
$$\begin{aligned} q' &= \frac{(\epsilon_1 - \epsilon_2)}{(\epsilon_1 + \epsilon_2)} q \\ q'' &= \frac{2\epsilon_2}{(\epsilon_1 + \epsilon_2)} q \end{aligned} \quad (4.54)$$

We can now obtain the potential which is given by

$$\Phi_1(z) = \frac{1}{4\pi\epsilon_1} \left( \frac{q}{\sqrt{s^2 + (d - z)^2}} + \frac{(\epsilon_1 - \epsilon_2)}{(\epsilon_1 + \epsilon_2)} \frac{q}{\sqrt{s^2 + (d + z)^2}} \right), \quad z > 0. \quad (4.55)$$

$$\Phi_2(z) = \frac{1}{4\pi\epsilon_2} \frac{2\epsilon_2}{(\epsilon_1 + \epsilon_2)} \frac{q}{\sqrt{s^2 + (d - z)^2}}, \quad z < 0. \quad (4.56)$$

For the two cases  $\epsilon_2 > \epsilon_1$  and  $\epsilon_2 < \epsilon_1$  the field lines of  $\mathbf{D}$  are shown qualitatively in Fig.4.11.



**Fig.4.11.** Lines of electric displacement for a point charge embedded in a dielectric 1 near a semi-infinite slab of dielectric 2.

The polarization-charge density is given by  $\rho_p = -\nabla \cdot \mathbf{P} = -\nabla \cdot [(1 - \varepsilon_0 / \varepsilon)\mathbf{D}]$ . In the regions where  $\varepsilon$  is constant it can be taken out of differentiation which leads to  $\rho_p = -(1 - \varepsilon_0 / \varepsilon)\nabla \cdot \mathbf{D} = 0$ , where we took into account the Gauss law. Thus, for a linear and uniform dielectric, in the absence of volume free charges, it must be no volume polarization charge. In our problem the polarization charge will be only at the point of charge  $q$ . At the interface between the two dielectrics, however,  $\varepsilon$  takes a discontinuous jump, as  $z$  passes through  $z = 0$ . This implies that there is a polarization-surface-charge density on the plane  $z = 0$ . The latter can be calculated by generalizing eq.(4.12) to the case of two dielectrics:

$$\sigma_p = (\mathbf{P}_1 - \mathbf{P}_2) \cdot \mathbf{n}. \quad (4.57)$$

where  $\mathbf{n} = -\hat{\mathbf{z}}$  is the unit normal pointing from dielectric 1 to dielectric 2, and  $\mathbf{P}_i$  ( $i=1,2$ ) is the polarization in the dielectric  $i$  at  $z = 0$ . Since

$$\mathbf{P}_i = (\varepsilon_i - \varepsilon_0)\mathbf{E}_i = -(\varepsilon_i - \varepsilon_0)\nabla \cdot \Phi_i|_{z=0}, \quad (4.58)$$

we find from eqs. (4.55) and (4.56) that

$$P_{1z} = -(\varepsilon_1 - \varepsilon_0) \frac{d\Phi_1}{dz} \Big|_{z=0} = -(\varepsilon_1 - \varepsilon_0) \frac{q}{4\pi\varepsilon_1} \left\{ \frac{(d-z)}{[s^2 + (d-z)^2]^{3/2}} - \frac{(\varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 + \varepsilon_2)} \frac{(d+z)}{[s^2 + (d+z)^2]^{3/2}} \right\} \Big|_{z=0} =$$

$$-\frac{q}{2\pi} \frac{\varepsilon_2(\varepsilon_1 - \varepsilon_0)}{\varepsilon_1(\varepsilon_1 + \varepsilon_2)} \frac{d}{[s^2 + d^2]^{3/2}} \quad (4.59)$$

$$P_{2z} = -(\varepsilon_2 - \varepsilon_0) \frac{d\Phi_2}{dz} \Big|_{z=0} = -(\varepsilon_2 - \varepsilon_0) \frac{q}{4\pi\varepsilon_2} \frac{2\varepsilon_2}{(\varepsilon_1 + \varepsilon_2)} \frac{(d-z)}{[s^2 + (d-z)^2]^{3/2}} \Big|_{z=0} = -\frac{q}{2\pi} \frac{(\varepsilon_2 - \varepsilon_0)}{(\varepsilon_1 + \varepsilon_2)} \frac{d}{[s^2 + d^2]^{3/2}}. \quad (4.60)$$

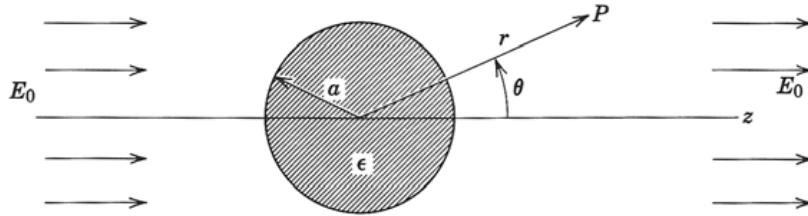
Therefore,

$$\sigma_p = -P_{1z} + P_{2z} = \frac{q}{2\pi} \frac{\varepsilon_0(\varepsilon_1 - \varepsilon_2)}{\varepsilon_1(\varepsilon_1 + \varepsilon_2)} \frac{d}{[s^2 + d^2]^{3/2}}. \quad (4.61)$$

In the limit  $\epsilon_1 = \epsilon_0$  and  $\epsilon_2 \rightarrow \infty$  the electric field inside dielectric 2 becomes very small and hence it behaves much like a conductor. Then the surface-charge density (4.61) approaches the value appropriate to a conducting surface,

$$\sigma_p = -\frac{q}{2\pi} \frac{d}{[s^2 + d^2]^{3/2}}. \quad (4.62)$$

**Example 2.** The second illustration of electrostatic problems involving dielectrics is that of a dielectric sphere of radius  $a$  with dielectric constant  $\epsilon_r = \epsilon / \epsilon_0$  placed in a uniform electric field, which at large distances from the sphere is directed along the  $z$  axis and has magnitude  $E_0$ , as indicated in Fig.4.12.



**Fig.4.12**

Both inside and outside the sphere there are no free charges. Consequently the problem is one of solving the Laplace equation with the proper boundary conditions at  $r = a$ . From the axial symmetry of the geometry we can take the solution to be of the form a Legendre polynomial expansion. The solution is different for the region inside and outside the sphere and take the form:

$$\Phi_{in}(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta), \quad r < a \quad (4.63)$$

$$\Phi_{out}(r, \theta) = \sum_{l=0}^{\infty} (B_l r^l + C_l r^{-l-1}) P_l(\cos \theta), \quad r > a \quad (4.64)$$

The coefficients  $A_l$ ,  $B_l$  and  $C_l$  are to be determined from the boundary conditions. From the boundary condition at infinity suggesting that

$$\Phi \rightarrow -E_0 r \cos \theta, \quad r \gg a \quad (4.65)$$

we find that the only non-vanishing  $B_l$  is  $B_1 = -E_0$ . The other coefficients are determined from the boundary conditions at  $r = a$ . The electrostatic potential should be continuous on the surface and hence

$$\Phi_{in}|_{r=a} = \Phi_{out}|_{r=a}. \quad (4.66)$$

Since there are no free surface charge the normal component of the electric displacement should also be continuous on the surface and hence

$$-\epsilon \frac{\partial \Phi_{in}}{\partial r} \Big|_{r=a} = -\epsilon_0 \frac{\partial \Phi_{out}}{\partial r} \Big|_{r=a}. \quad (4.67)$$

Now we substitute the series (4.63) and (4.64) in these boundary conditions. For the first boundary condition (4.63) this leads

$$\sum_{l=0}^{\infty} A_l a^l P_l(\cos \theta) = \sum_{l=0}^{\infty} \left( -E_0 a^l \delta_{l1} + C_l \frac{1}{a^{l+1}} \right) P_l(\cos \theta). \quad (4.68)$$

Since this equation must be valid for all  $\theta$ , the coefficient of each Legendre function must be equal and therefore

$$\begin{aligned} A_1 &= -E_0 + \frac{C_1}{a^3} \\ A_l &= \frac{C_l}{a^{2l+1}}, \quad l \neq 1 \end{aligned} \quad (4.69)$$

The second boundary condition (4.64) leads to

$$\varepsilon \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos \theta) = \varepsilon_0 \sum_{l=0}^{\infty} \left( -E_0 \delta_{l1} - (l+1) C_l \frac{1}{a^{l+2}} \right) P_l(\cos \theta). \quad (4.70)$$

Since this equation must be valid for all  $\theta$ , the coefficient of each Legendre function must be equal and therefore

$$\begin{aligned} \frac{\varepsilon}{\varepsilon_0} A_1 &= -E_0 - \frac{2C_1}{a^3} \\ \frac{\varepsilon}{\varepsilon_0} l A_l &= -(l+1) \frac{C_l}{a^{2l+1}}, \quad l \neq 1 \end{aligned} \quad (4.71)$$

The second equations in (4.69) and (4.71) can be satisfied simultaneously only with  $A_l = C_l = 0$  for all  $l \neq 1$ . The remaining coefficients are given in terms of the applied electric field  $E_0$

$$\begin{aligned} A_1 &= -\frac{3}{\varepsilon_r + 2} E_0 \\ C_1 &= \frac{\varepsilon_r - 1}{\varepsilon_r + 2} a^3 E_0 \end{aligned} \quad (4.72)$$

The potential is therefore

$$\Phi_m(r, \theta) = -\frac{3}{\varepsilon_r + 2} E_0 r \cos \theta, \quad r < a \quad (4.73)$$

$$\Phi_{out}(r, \theta) = -E_0 r \cos \theta + \frac{\varepsilon_r - 1}{\varepsilon_r + 2} E_0 \frac{a^3}{r^2} \cos \theta, \quad r > a \quad (4.74)$$

It is easy to see that the potential inside the sphere describes a constant electric field parallel to the applied field with magnitude

$$\mathbf{E}_m = \frac{3}{\varepsilon_r + 2} \mathbf{E}_0. \quad (4.75)$$

Outside the sphere the potential is equivalent to the applied field  $\mathbf{E}_0$  plus the field of an electric dipole at the origin with dipole moment:

$$\mathbf{p} = 4\pi\varepsilon_0 \frac{\varepsilon_r - 1}{\varepsilon_r + 2} a^3 \mathbf{E}_0, \quad (4.76)$$

oriented in the direction of the applied field.

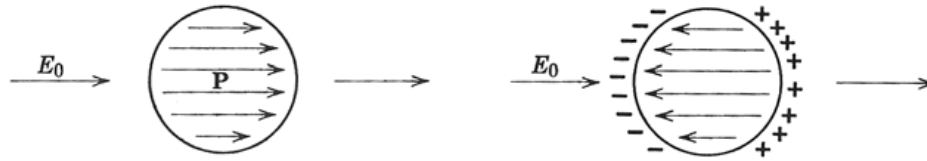
The dipole moment can be interpreted as the volume integral of the polarization  $\mathbf{P}$ . The polarization is

$$\mathbf{P} = (\epsilon - \epsilon_0)\mathbf{E} = 3\epsilon_0 \frac{(\epsilon_r - 1)}{\epsilon_r + 2} \mathbf{E}_0. \quad (4.77)$$

It is constant throughout the volume of the sphere and has a volume integral given by (4.76). The polarization-surface-charge density is

$$\sigma_p = \mathbf{P} \cdot \mathbf{n} = 3\epsilon_0 \frac{(\epsilon_r - 1)}{\epsilon_r + 2} E_0 \cos \theta. \quad (4.78)$$

This can be thought of as producing an internal field directed oppositely to the applied field, so reducing the field inside the sphere to its value (4.75), as sketched in Fig. 4.13.



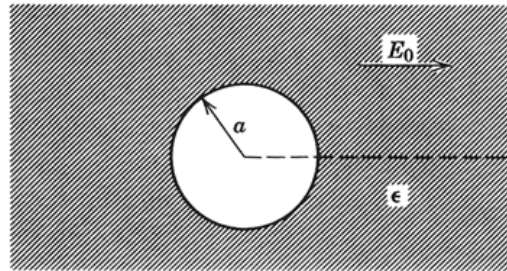
**Fig. 4.13** Dielectric sphere in a uniform field  $E_0$ , showing the polarization on the left and the polarization charge with its associated, opposing, electric field on the right.

The problem of a spherical cavity of radius  $a$  in a dielectric medium with dielectric constant  $\epsilon_r = \epsilon / \epsilon_0$  and with an applied electric field  $\mathbf{E}_0$  parallel to the  $z$  axis, as shown in Fig.4.14, can be handled in exactly the same way as the dielectric sphere. In fact, inspection of boundary conditions (4.66) and (4.67) shows that the results for the cavity can be obtained from those of the sphere by the replacement  $\epsilon / \epsilon_0$  to  $\epsilon_0 / \epsilon$ . Thus, for example, the field inside the cavity is uniform, parallel to  $\mathbf{E}_0$ , and of magnitude:

$$\mathbf{E}_{in} = \frac{3\epsilon_r}{2\epsilon_r + 1} \mathbf{E}_0. \quad (4.79)$$

Similarly, the field outside is the applied field plus that of a dipole at the origin *oriented oppositely* to the applied field and with dipole moment

$$\mathbf{p} = -4\pi\epsilon_0 \frac{\epsilon_r - 1}{2\epsilon_r + 1} a^3 \mathbf{E}_0. \quad (4.80)$$



**Fig.4.14.** Spherical cavity in a dielectric with a uniform field applied.

## Electrostatic energy in dielectrics

In free space we derived the energy of a distribution of charge  $\rho(\mathbf{r})$  by assembling the distribution little by little, bringing infinitesimal pieces of charge in from infinity. Following this reasoning we found that

$$U = \frac{1}{2} \int \rho(\mathbf{r}) \Phi(\mathbf{r}) d^3 r . \quad (4.81)$$

This is in general not true in the presence of dielectrics (however, as we will see, it may be true in some cases). In the presence of dielectrics work must also be done to induce polarization in the dielectric, and it is not clear if this work is included in the equation above.

When dielectrics are present we will use a somewhat different argument (which still corresponds to the same procedure). Suppose that there is initially some macroscopic charge density  $\rho(\mathbf{r})$ , potential  $\Phi(\mathbf{r})$ , and fields  $\mathbf{E}(\mathbf{r})$  and  $\mathbf{D}(\mathbf{r})$ . Imagine that some infinitesimal change in the charge density,  $\delta\rho(\mathbf{r})$ , is made. To first order in  $\delta\rho(\mathbf{r})$ , the change in energy of the system is

$$\delta U = \int \delta\rho(\mathbf{r}) \Phi(\mathbf{r}) d^3 r , \quad (4.82)$$

where the integration is performed over all space. The point is that this is the interaction energy of  $\delta\rho(\mathbf{r})$  with the sources already present (and which produce  $\Phi(\mathbf{r})$ ); the interaction energy of  $\delta\rho(\mathbf{r})$  with itself is second-order in small (infinitesimal) quantities.

The change in  $\mathbf{D}$  which arises as a consequence of the change  $\delta\rho(\mathbf{r})$  in the charge density is related to the latter by the equation  $\nabla \cdot (\mathbf{D} + \delta\mathbf{D}) = \rho + \delta\rho$  and therefore  $\delta\rho = \nabla \cdot \delta\mathbf{D}$ , where  $\delta\mathbf{D}$  is the resulting change in  $\mathbf{D}$ , so we can write the change in the energy as

$$\delta U = \int (\nabla \cdot \delta\mathbf{D}) \Phi d^3 r , \quad (4.83)$$

Now

$$\nabla \cdot (\delta\mathbf{D}\Phi) = (\nabla \cdot \delta\mathbf{D})\Phi + \delta\mathbf{D} \cdot (\nabla\Phi) = (\nabla \cdot \delta\mathbf{D})\Phi - \delta\mathbf{D} \cdot \mathbf{E} , \quad (4.84)$$

and hence integrating by parts we obtain

$$\delta U = \int \nabla \cdot (\delta\mathbf{D}\Phi) d^3 r + \int \delta\mathbf{D} \cdot \mathbf{E} d^3 r . \quad (4.85)$$

The divergence theorem turns the first term into a surface integral, which vanishes for a localized charge distribution if the integration is performed over all space:

$$\int_V \nabla \cdot (\delta\mathbf{D}\Phi) d^3 r = \oint_S (\delta\mathbf{D}\Phi) \cdot \mathbf{n} da = 0 . \quad (4.86)$$

Therefore, the work done is equal to

$$\delta U = \int \delta\mathbf{D} \cdot \mathbf{E} d^3 r . \quad (4.87)$$

So far it implies to *any* material. Now, if the material is a linear dielectric, then  $\mathbf{D} = \varepsilon\mathbf{E}$ , so infinitesimal increments

$$\frac{1}{2} \delta(\mathbf{D} \cdot \mathbf{E}) = \frac{1}{2} \delta(\varepsilon E^2) = \varepsilon \delta\mathbf{E} \cdot \mathbf{E} = \delta\mathbf{D} \cdot \mathbf{E} . \quad (4.88)$$

Thus the change in the electrostatic energy is

$$\delta U = \frac{1}{2} \int \delta(\mathbf{D} \cdot \mathbf{E}) d^3 r. \quad (4.89)$$

Now we build the free charge up from zero to the final configuration. This corresponds to integrating from zero field up to the final field (a functional integration),

$$U = \frac{1}{2} \int d^3 r \int_0^{\mathbf{D}} \delta(\mathbf{D} \cdot \mathbf{E}). \quad (4.90)$$

This leads to

$$U = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} d^3 r. \quad (4.91)$$

This result is valid only for linear media and leads to eq. (4.81). Indeed, using  $\mathbf{E} = -\nabla\Phi$  and integrating (4.91) by parts, we obtain

$$U = -\frac{1}{2} \int \mathbf{D} \cdot \nabla\Phi d^3 r = -\frac{1}{2} \int \nabla \cdot (\mathbf{D}\Phi) d^3 r + \frac{1}{2} \int \Phi \nabla \cdot \mathbf{D} d^3 r. \quad (4.92)$$

Through the divergence theorem, the first term yields a surface term which vanishes at infinity. The second term becomes

$$U = \frac{1}{2} \int \Phi \nabla \cdot \mathbf{D} d^3 r = \frac{1}{2} \int \rho(\mathbf{r}) \Phi(\mathbf{r}) d^3 r. \quad (4.93)$$

Thus for a linear dielectric, the original formula is valid.