

## Section 2: Electrostatics

### Uniqueness of solutions of the Laplace and Poisson equations

If electrostatics problems always involved localized discrete or continuous distribution of charge with no boundary conditions, the general solution for the potential

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r', \quad (2.1)$$

would be the most convenient and straightforward solution to any problem. There would be no need of the Poisson or Laplace equations. In actual fact, of course, many, if not most, of the problems of electrostatics involve finite regions of space, with or without charge inside, and with prescribed boundary conditions on the bounding surfaces. These boundary conditions may be simulated by an appropriate distribution of charges outside the region of interest and eq.(2.1) becomes inconvenient as a means of calculating the potential, except in simple cases such as method of images.

In general case we have to solve the Poisson or Laplace equation depending on the presence of the charge density in the region of consideration. The Poisson equation

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \quad (2.2)$$

and the Laplace equation

$$\nabla^2\Phi = 0 \quad (2.3)$$

are linear, second order, partial differential equations. Therefore, to determine a solution we have also to specify boundary conditions.

*Example: One-dimensional problem*

$$\frac{d^2\Phi}{dx^2} = \lambda \quad (2.4)$$

for  $x \in [0, L]$ , where  $\lambda$  is a constant. The solution is

$$\Phi = \frac{1}{2}\lambda x^2 + ax + b, \quad (2.5)$$

where  $a$  and  $b$  are constants. To determine these constants, we might specify the values of  $\Phi(x=0)$  and  $\Phi(x=L)$ , i.e. the values on the boundary.

Now consider the solution of the Poisson equation within a finite volume  $V$ , bounded by a closed surface  $S$ . What boundary conditions are appropriate for the Poisson (or Laplace) equation to ensure that a unique and well-behaved (i.e., physically reasonable) solution will exist inside the bounded region  $V$ ? Physical experience leads us to believe that there are two possibilities.

(i) to specify the potential on a closed surface  $S$ , such that

$$\Phi(\mathbf{r})|_{\mathbf{r} \in S} = f(\mathbf{r}). \quad (2.6)$$

This is called *Dirichlet boundary condition*. An example is the electrostatic potential in a cavity inside a conductor, with the potential specified on the boundaries.

(ii) to specify the electric field (normal derivative of the potential) everywhere on the surface (corresponding to a given surface-charge density), such that

$$\mathbf{n} \cdot \nabla \Phi \Big|_{\mathbf{r} \in S} = \frac{\partial \Phi}{\partial n} \Big|_{\mathbf{r} \in S} = g(\mathbf{r}). \quad (2.7)$$

Specification of the normal derivative is known as the *Neumann boundary condition*. An example is electrostatic potential inside  $S$ , with charge on  $\sigma$  specified on the boundaries.

We will prove that the solutions of the Laplace and Poisson equations are unique if they are subject to either of the above boundary conditions.

To handle the boundary conditions we first derive useful identities known as Green's identities. These follow as simple applications of the divergence theorem. The divergence theorem states that

$$\int_V \nabla \cdot \mathbf{A} d^3 r = \oint_S \mathbf{A} \cdot \mathbf{n} da, \quad (2.8)$$

for any well-behaved vector field  $\mathbf{A}$  defined in the volume  $V$  bounded by the closed surface  $S$ . Let  $\mathbf{A} = \phi \nabla \psi$ , where  $\phi$  and  $\psi$  are arbitrary scalar fields. Now

$$\nabla \cdot \mathbf{A} = \nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi, \quad (2.9)$$

and

$$\mathbf{A} \cdot \mathbf{n} = \phi \nabla \psi \cdot \mathbf{n} = \phi \frac{\partial \psi}{\partial n}, \quad (2.10)$$

where  $\partial/\partial n$  is the normal derivative at the surface  $S$  (directed outward from inside the volume  $V$ ). When (2.9) and (2.10) are substituted into the divergence theorem, there results *Green's first identity*:

$$\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d^3 r = \oint_S \phi \frac{\partial \psi}{\partial n} da. \quad (2.11)$$

If we write down (2.11) again with  $\phi$  and  $\psi$  interchanged, and then subtract it from (2.11), the  $\nabla \phi \cdot \nabla \psi$  terms cancel, and we obtain Green's second identity or *Green's theorem*

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) d^3 r = \oint_S \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) da. \quad (2.12)$$

Now let us show the uniqueness of the solution of the Poisson equation,  $\nabla^2 \Phi = -\rho/\epsilon_0$ , inside a volume  $V$  subject to either Dirichlet or Neumann boundary conditions on the closed bounding surface  $S$ . We suppose, to the contrary, that there exist two solutions  $\Phi_1$  and  $\Phi_2$  satisfying the same boundary conditions, either

(i)  $\Phi_{1,2}(\mathbf{r}) \Big|_{\mathbf{r} \in S} = f(\mathbf{r})$  for the Dirichlet boundary condition,

(ii)  $\frac{\partial \Phi_{1,2}}{\partial n} \Big|_{\mathbf{r} \in S} = g(\mathbf{r})$  for the Neumann boundary condition.

where  $f(\mathbf{r})$  and  $g(\mathbf{r})$  are continuous functions defined on surface  $S$ . Consider the function

$$U(\mathbf{r}) = \Phi_1(\mathbf{r}) - \Phi_2(\mathbf{r}) \quad (2.13)$$

Then  $U(\mathbf{r})$  satisfies the Laplace equation

$$\nabla^2 U(\mathbf{r}) = 0 \quad (2.14)$$

inside  $V$ , and

(i)  $U(\mathbf{r})|_{\mathbf{r} \in S} = 0$  for the Dirichlet boundary condition,

(ii)  $\frac{\partial U}{\partial n}|_{\mathbf{r} \in S} = 0$  for the Neumann boundary condition.

From Green's first identity (2.11), with  $\phi = \psi = U$ , we find

$$\int_V (U \nabla^2 U + \nabla U \cdot \nabla U) d^3 r = \oint_S U \frac{\partial U}{\partial n} da. \quad (2.15)$$

Taking into account the fact the  $\nabla^2 U = 0$  in volume  $V$  and applying the boundary conditions [either (i) or (ii)], this reduces to

$$\int_V |\nabla U|^2 d^3 r = 0. \quad (2.16)$$

Since  $|\nabla U|^2$  is a *positive indefinite* for all  $\mathbf{r} \in V$ , eq.(2.16) implies that  $\nabla U = 0$  everywhere in volume  $V$  and hence,  $U = \text{const}$  in  $V$ . Therefore, we find that

(i) For Dirichlet boundary condition,  $U = 0$  on surface  $S$  so that, inside  $V$ ,  $\Phi_1 = \Phi_2$  and the solution is unique.

(ii) For Neumann boundary conditions,  $\partial U / \partial n = 0$  on the surface, the solution is unique, apart from an unimportant arbitrary additive constant.

Some observations follow from on the proof:

(i) We can specify either Dirichlet or Neumann boundary conditions at each point on the boundary, but not both. To specify both is inconsistent, since the solution is then *overdetermined*.

(ii) However, we can specify either Dirichlet or Neumann boundary conditions on different parts of the surface.

(iii) The uniqueness property means we can use any method we wish to obtain the solution - if it satisfies the correct boundary conditions, and is a solution of the equation, then it is the correct solution. A good example is method of images which we will consider in the next chapter.

### **Formal solution of electrostatic boundary-value problem. Green's function.**

The solution of the Poisson or Laplace equation in a finite volume  $V$  with either Dirichlet or Neumann boundary conditions on the bounding surface  $S$  can be obtained by means of so-called Green's functions.

The simplest example of Green's function is the Green's function of free space:

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \quad (2.17)$$

Using this Green's function the solution of electrostatic problem with a known localized charge distribution can be written as follows:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r' = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') G_0(\mathbf{r}, \mathbf{r}') d^3 r'. \quad (2.18)$$

A Green's function of free space  $G_0(\mathbf{r}, \mathbf{r}')$  gives the response at point  $\mathbf{r}$ , due to a unit point charge placed at  $\mathbf{r}'$ . For the response due to a collection of charges characterized by charge density  $\rho(\mathbf{r}')$ , superposition yields the integral above. The Poisson equation applied to the potential of a point source says that

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta^3(\mathbf{r}-\mathbf{r}'). \quad (2.19)$$

Therefore, the Green's function of free space obeys differential equation

$$\nabla^2 G_0(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}'). \quad (2.20)$$

For the more general problem of a collection of charges and boundary conditions on a surface, we might expect that the solution for the potential (2.18) should be generalized to include surface term so that

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') G_0(\mathbf{r},\mathbf{r}') d^3 r' + \text{surface term}. \quad (2.21)$$

In order to show this explicitly let us use Green's second identity

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) d^3 r' = \oint_S \left( \phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'} \right) da', \quad (2.22)$$

assuming that  $\phi = \Phi$  and  $\psi = G_0(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}$ , where  $\mathbf{r}$  is the observation point and  $\mathbf{r}'$  is the integration variable. Taking into account eq.(2.19) and the Poisson equation  $\nabla^2 \Phi = -\rho/\epsilon_0$  we find:

$$\int_V \left[ -4\pi\Phi(\mathbf{r}')\delta^3(\mathbf{r}-\mathbf{r}') + \frac{\rho(\mathbf{r}')}{\epsilon_0|\mathbf{r}-\mathbf{r}'|} \right] d^3 r' = \oint_S \left[ \Phi(\mathbf{r}') \frac{\partial}{\partial n'} \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial \Phi(\mathbf{r}')}{\partial n'} \right] da', \quad (2.23)$$

For the point  $\mathbf{r}$  lying within volume  $V$  we obtain:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3 r' + \frac{1}{4\pi} \oint_S \left[ \frac{1}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial \Phi(\mathbf{r}')}{\partial n'} - \Phi(\mathbf{r}') \frac{\partial}{\partial n'} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \right] da'. \quad (2.24)$$

It is easy to see that if the surface  $S$  goes to infinity and the potential on  $S$  falls off  $R^{-1}$  or faster, then the surface integral vanishes and (2.24) reduces to the familiar result (2.18).

It is important to note that for a charge-free volume, the potential anywhere inside the volume (a solution of the Laplace equation) is expressed in (2.24) in terms of the potential and its normal derivative only on the surface of the volume. This rather surprising result is *not* a solution to a boundary-value problem, but only an integral statement, since the arbitrary specification of both the potential and its derivative on the surface is *overspecification* of the problem.

It is clear therefore that our integral expression for  $\Phi(\mathbf{r})$  which involves surface integrals of both the potential and its normal derivative, is not a very effective way to solve an electrostatic boundary value problem; it requires more input information than is actually needed to determine a solution and hence represents an integral equation for the potential. If we had made a better choice of  $\psi(\mathbf{r})$  at the outset, we could have come up with a better result. Let's try again, choosing for  $\psi(\mathbf{r})$  a function we will call

Green's function  $G(\mathbf{r},\mathbf{r}')$ ; it is given by  $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$  plus an as yet undetermined function  $F(\mathbf{r},\mathbf{r}')$ :

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} + F(\mathbf{r},\mathbf{r}'). \quad (2.25)$$

We impose the condition that a Green's function satisfies the equation

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta^3(\mathbf{r} - \mathbf{r}'). \quad (2.26)$$

which implies that  $F(\mathbf{r}, \mathbf{r}')$  is a solution of the Laplace equation in  $V$ ,

$$\nabla^2 F(\mathbf{r}, \mathbf{r}') = 0. \quad (2.27)$$

With the generalized concept of a Green's function and its additional freedom via the function  $F(\mathbf{r}, \mathbf{r}')$  we can choose  $F(\mathbf{r}, \mathbf{r}')$  to eliminate one or the other of the two surface integrals, obtaining a result that involves *only* Dirichlet or Neumann boundary conditions. Of course, if the necessary  $G(\mathbf{r}, \mathbf{r}')$  depended in detail on the exact form of the boundary conditions, the method would have little generality. As will be seen immediately, this is not required, and  $G(\mathbf{r}, \mathbf{r}')$  satisfies rather simple boundary conditions on  $S$ .

With Green's theorem (2.22),  $\phi = \Phi$  and  $\psi = G(\mathbf{r}, \mathbf{r}')$ , and the specified properties of  $G$  (2.26), it is simple to obtain the generalization of (2.24):

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') d^3 r' + \frac{1}{4\pi} \oint_S \left[ G(\mathbf{r}, \mathbf{r}') \frac{\partial \Phi(\mathbf{r}')}{\partial n'} - \Phi(\mathbf{r}') \frac{\partial}{\partial n'} G(\mathbf{r}, \mathbf{r}') \right] da'. \quad (2.28)$$

The freedom available in the definition of  $G$  (2.25) means that we can make the surface integral depend only on the chosen type of boundary conditions. Thus, for *Dirichlet boundary conditions* we demand:

$$G_D(\mathbf{r}, \mathbf{r}') = 0 \quad \text{for } \mathbf{r}' \in S. \quad (2.29)$$

Then the first term in the surface integral in (2.28) vanishes and the solution is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') G_D(\mathbf{r}, \mathbf{r}') d^3 r' - \frac{1}{4\pi} \oint_S \Phi(\mathbf{r}') \frac{\partial}{\partial n'} G_D(\mathbf{r}, \mathbf{r}') da'. \quad (2.30)$$

We have all the necessary information to solve a Dirichlet problem with potential  $\Phi(\mathbf{r}')$  specified on surface  $S$  in terms of Green's function  $G_D(\mathbf{r}, \mathbf{r}')$  which is the solution of eq. (2.26) subject to boundary conditions (2.29). This function exists and it is unique. The preceding two conditions are sufficient to determine it completely. We know this without resorting to a mathematical proof because the  $G_D(\mathbf{r}, \mathbf{r}')$  is just the scalar potential at point  $\mathbf{r}$  given a unit point charge at point  $\mathbf{r}'$  inside of a cavity with conducting walls coincident with  $S$  held at zero potential. This is the physical interpretation of the Dirichlet Green's function. Notice that this is a strongly geometry-dependent function but it is not dependent on any other properties of the system. In other words, we can solve any Dirichlet problem for a given geometry if we can solve the problem of a point charge with grounded conducting surfaces. An important property of the Dirichlet Green's function is that  $G_D(\mathbf{r}, \mathbf{r}') = G_D(\mathbf{r}', \mathbf{r})$ .

For *Neumann boundary conditions* we must be more careful. The obvious choice of boundary condition on  $G_N(\mathbf{r}, \mathbf{r}')$  seems to be

$$\frac{\partial G_N(\mathbf{r}, \mathbf{r}')}{\partial n'} = 0 \quad \text{for } \mathbf{r}' \in S. \quad (2.31)$$

since that makes the second term in the surface integral in (2.28) vanish, as desired. But an application of Gauss's theorem to (2.26) shows that

$$-4\pi = \int_V \nabla \cdot [\nabla G(\mathbf{r}, \mathbf{r}')] d^3 r' = \oint_S \mathbf{n}' \cdot \nabla G(\mathbf{r}, \mathbf{r}') da' = \oint_S \frac{\partial}{\partial n'} G(\mathbf{r}, \mathbf{r}') da', \quad (2.32)$$

which means that  $\partial G(\mathbf{r}, \mathbf{r}')/\partial n'$  is not zero *everywhere* on the surface  $S$ . Consequently the simplest allowable boundary condition on  $G_N$  is

$$\frac{\partial G_N(\mathbf{r}, \mathbf{r}')}{\partial n'} = -\frac{4\pi}{S} \quad \text{for } \mathbf{r}' \in S. \quad (2.33)$$

where  $S$  is the total area of the boundary surface. Then the solution is

$$\Phi(\mathbf{r}) = \langle \Phi \rangle_S + \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') G_N(\mathbf{r}, \mathbf{r}') d^3 r' + \frac{1}{4\pi} \oint_S \frac{\partial \Phi(\mathbf{r}')}{\partial n'} G_N(\mathbf{r}, \mathbf{r}') da'. \quad (2.34)$$

Where  $\langle \Phi \rangle_S$  is the average value of the potential over the whole surface:

$$\langle \Phi \rangle_S = \frac{1}{S} \oint_S \Phi(\mathbf{r}') da'. \quad (2.35)$$

One can understand the necessity of the presence of this term from the fact that the Neumann boundary condition problem can only be solved up to an arbitrary constant.

### Method of images

Boundary-value problems in electrostatics that involve point charges in the presence of boundary surfaces, e.g., provided by conductors at either zero or non-zero *fixed* potential, can sometimes be solved without explicit recourse to Green's function method. Indeed, when the geometry of a boundary surface is sufficiently simple, the related boundary value problem can be treated as though the boundary was not present at all by *simulating* the boundary conditions by a small number of (point) charges of appropriate magnitude, suitably placed at positions *external* to the region of interest. The distribution of these so-called *image charges* must be of such a nature that their individual electrostatic scalar potentials add up of to form an equipotential surface of the same shape and of the same value as the original boundary surface. The replacement of the original boundary value problem by a larger field region with image charges but *no* boundaries is called the *method of images*.

Thus we consider two distinct electrostatics problems. The first is the “real” problem in which we are given a charge density  $\rho(\mathbf{r})$  in volume  $V$  and some boundary conditions on the surface  $S$  bounding this volume. The second is a “fictitious problem” in which the charge density inside of  $V$  is the same as for the real problem and in which there is some undetermined charge distribution elsewhere. The latter is to be chosen in such a way that the solution to the second problem satisfies the boundary conditions specified in the first problem. Then the solution to the second problem is also the solution to the first problem inside of volume  $V$  (but not outside of  $V$ !). If one has found the initially undetermined exterior charge in the second problem, called image charge, then the potential is found simply from Coulomb's law,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_2(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r'. \quad (2.36)$$

where  $\rho_2(\mathbf{r}')$  is the total charge density of the second problem.

### Point charge near a grounded conducting plane

Consider a simple example. Suppose that we have a point charge  $q$  located at a point  $\mathbf{r}_0 = (0, 0, a)$  in Cartesian coordinates. A grounded conductor occupies the half-space  $z < 0$ , which means that we have the Dirichlet boundary condition at  $z = 0$ :  $\Phi(x, y, 0) = 0$ . We can also assume that  $\Phi(x, y, \infty) = 0$ . First, we need to determine image charge located in the half-space  $z < 0$  such that the image charge and the real charge produce zero potential on the  $z = 0$  plane. It is easy to realize that a single image charge  $-q$  located at the point  $\mathbf{r}'_0 = -\mathbf{r}_0 = (0, 0, -a)$  is what is required. Since all points on the  $z = 0$  plane are equidistant from the real charge and its image, the two charges produce canceling potentials at these points. The solution to the problem is therefore

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{|\mathbf{r}-\mathbf{r}_0|} - \frac{1}{|\mathbf{r}+\mathbf{r}_0|} \right) = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{\sqrt{(a-z)^2 + s^2}} - \frac{1}{\sqrt{(a+z)^2 + s^2}} \right]. \quad (2.37)$$

where  $s = \sqrt{x^2 + y^2}$ . This function satisfies the correct Poisson equation in the domain  $z > 0$  and also satisfies the correct boundary conditions at  $z = 0$ ; therefore it is the *unique* solution of the electrostatics problem. It is important to realize, however, that this solution is incorrect in the space  $z < 0$ ; here, the real potential is zero because this domain is inside of the grounded conductor, whereas the potential (2.37) is not zero.

Obviously in the real system, the real charge  $q$  produces a surface charge density  $\sigma(x, y)$  on the conductor. To determine it, we have to evaluate the normal component of the electric field at the surface of the conductor:

$$E_n(x, y, 0) = - \left. \frac{d\Phi(x, y, z)}{dz} \right|_{z=0} = - \frac{qa}{2\pi\epsilon_0 (a^2 + s^2)^{3/2}}. \quad (2.38)$$

The surface charge density is, therefore,

$$\sigma(x, y, 0) = \epsilon_0 E_n(x, y, 0) = - \frac{qa}{2\pi (a^2 + s^2)^{3/2}}. \quad (2.39)$$

From this example we can also see why this technique is called “method of images”. The image charge is precisely the mirror image with respect to the  $z = 0$  plane of the real charge.

The total induced surface charge can be found by integrating (2.39) over the plane. This result can however be obtained easier from the Gauss law. By choosing an infinite closed surface that includes both the plane and the charge we immediately obtain

$$q = -\epsilon_0 \oint_S \sigma da. \quad (2.40)$$

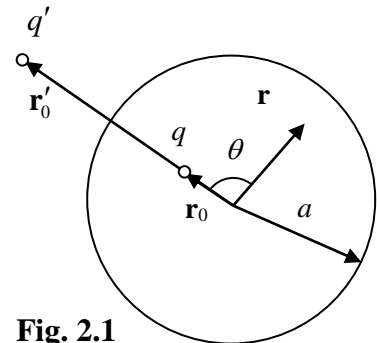
From the above consideration we can conclude that the Dirichlet Green’s function for the semi-infinite half-space  $z > 0$  is

$$G(\mathbf{r}, \mathbf{r}') = \left( \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'_i|} \right). \quad (2.41)$$

where  $\mathbf{r}'_i = (x', y', -z')$  is the mirror image of  $\mathbf{r}' = (x', y', z')$  with respect to the  $z = 0$  plane. Hence, by performing appropriate integration we can find the solution of electrostatic problem for any given charge density  $\rho(\mathbf{r})$  in the domain  $z > 0$  and an arbitrary potential  $\Phi(x, y, 0)$  on the surface.

### Point Charge in a Spherical Cavity

In some cases it is also possible to use the method of images when the boundary  $S$  is a curved surface. However, just as curved mirrors produced distorted images, curved surfaces make the image of a point charge more complicated. The simplest problem of this kind is a point charge in a spherical cavity (Fig.2.1). Suppose that we have a spherical cavity of radius  $a$  inside of a conductor. Within this cavity a point charge  $q$  is located at a distance  $r_0$  from the center of the sphere which is also chosen to be the origin of coordinates. Thus the charge is at point  $\mathbf{r}_0 = r_0 \mathbf{n}_0$ , where  $\mathbf{n}_0$  is a unit vector pointing in the direction



**Fig. 2.1**

from the origin to the charge. We need to find the image(s) of the charge in the spherical surface which encloses it. The simplest possible set of images would be a single charge  $q'$ ; if there is such a solution, symmetry considerations tell us that the image must be located on the line passing through the origin and going in the direction of  $\mathbf{n}_0$ . Let us therefore put an image charge  $q'$  at point  $\mathbf{r}'_0 = r'_0 \mathbf{n}_0$ .

The potential produced by the real and image charges is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{|\mathbf{r} - \mathbf{r}_0|} + \frac{q'}{|\mathbf{r} - \mathbf{r}'_0|} \right). \quad (2.42)$$

Now we must choose, if possible,  $q'$  and  $r'_0$  such that  $\Phi(\mathbf{r}) = 0$  for  $\mathbf{r}$  lying on the cavity's spherical surface,  $\mathbf{r} = a\mathbf{n}$  where the direction of  $\mathbf{n}$  is arbitrary. The potential at such a point may be written as

$$\Phi(a\mathbf{n}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{|a\mathbf{n} - r_0\mathbf{n}_0|} + \frac{q'}{|a\mathbf{n} - r'_0\mathbf{n}_0|} \right] = \frac{1}{4\pi\epsilon_0} \left[ \frac{q/a}{|\mathbf{n} - (r_0/a)\mathbf{n}_0|} + \frac{q'/r'_0}{|\mathbf{n}_0 - (a/r'_0)\mathbf{n}|} \right]. \quad (2.43)$$

It is clear that denominators are equal if  $r_0/a = a/r'_0$ . This is because  $\mathbf{n}$  and  $\mathbf{n}_0$  are unit vectors and consequently  $|\mathbf{n} - \alpha\mathbf{n}_0| = |\mathbf{n}_0 - \alpha\mathbf{n}|$  for arbitrary  $\alpha$  (which follows, e.g., from the cosine rule). The numerators are equal and opposite if  $q/a = -q'/r'_0$ . Hence, we make the potential  $\Phi(\mathbf{r})$  zero on the spherical surface by choosing

$$r'_0 = a^2 / r_0; \quad (2.44)$$

$$q' = -qr'/a = -q(a/r_0). \quad (2.45)$$

Thus we have obtained the solution for a point charge in a spherical cavity with an equipotential surface:

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{|\mathbf{r} - \mathbf{r}_0|} - \frac{a/r_0}{|\mathbf{r} - (a^2/r_0^2)\mathbf{r}_0|} \right). \quad (2.46)$$

We have also found the Dirichlet Green's function for the interior of a sphere of radius  $a$ :

$$G(\mathbf{r}, \mathbf{r}') = \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{a/r'}{|\mathbf{r} - (a^2/r'^2)\mathbf{r}'|} \right). \quad (2.47)$$

Let's look now at a few more features of the solution for the charge inside of the spherical cavity. First, we find the charge density on the surface of the cavity. From Gauss's law, we know that the charge density  $\sigma$  is the normal component of the electric field out of the conductor at its surface multiplied by  $\epsilon_0$ . This is the negative of the radial component in spherical polar coordinates, so

$$\sigma = -\epsilon_0 E_r = \epsilon_0 \left. \frac{\partial \Phi}{\partial r} \right|_{r=a}, \quad (2.48)$$

If we define the  $z$ -direction to be the direction of  $\mathbf{n}_0$ , then the potential at an arbitrary point within the sphere is

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{(r^2 + r_0^2 - 2rr_0 \cos \theta)^{1/2}} - \frac{a/r_0}{(r^2 + a^4/r_0^2 - 2r(a^2/r_0) \cos \theta)^{1/2}} \right]. \quad (2.49)$$



where  $\theta$  is the usual polar angle between the  $z$ -axis, or direction of  $\mathbf{n}_0$ , and the direction of  $\mathbf{r}$ , the point of observation. The radial component of the electric field at  $r = a$  is

$$E_r = \frac{q}{4\pi\epsilon_0} \left[ \frac{a - r_0 \cos \theta}{(a^2 + r_0^2 - 2ar_0 \cos \theta)^{3/2}} - \frac{a/r_0 (a - (a^2/r_0) \cos \theta)}{(a^2 + a^4/r_0^2 - 2a(a^2/r_0) \cos \theta)^{3/2}} \right] = \quad (2.50)$$

$$\frac{q}{4\pi\epsilon_0} \left[ \frac{a - r_0 \cos \theta}{(a^2 + r_0^2 - 2ar_0 \cos \theta)^{3/2}} - \frac{r_0^2/a(1 - (a/r_0) \cos \theta)}{(r_0^2 + a^2 - 2ar_0 \cos \theta)^{3/2}} \right] = \frac{q}{4\pi\epsilon_0 a^2} \frac{(1 - r_0^2/a^2)}{(1 + r_0^2/a^2 - 2(r_0/a) \cos \theta)^{3/2}}$$

If we introduce  $\delta = r_0/a$ , then the surface charge density may be written concisely as

$$\sigma = -\frac{q}{4\pi a^2} \frac{(1 - \delta^2)}{(1 + \delta^2 - 2\delta \cos \theta)^{3/2}}. \quad (2.51)$$

Notice that the surface charge density is largest in the direction of  $\mathbf{n}_0$  and is equal

$$\sigma_{\max} = -\frac{q}{4\pi a^2} \frac{(1 + \delta)}{(1 - \delta)^2}. \quad (2.52)$$

The total charge on the surface may be found by integrating  $\sigma$  over  $S$ . However, it may be obtained more easily by invoking Gauss's law. If we integrate the normal component of the electric field over a closed surface which lies entirely in conducting material and which also encloses the cavity, the integral will be zero, because the field in the conductor is zero:

$$0 = \oint_S E_n da = \frac{1}{\epsilon_0} \left[ q + \oint_S \sigma da \right]. \quad (2.53)$$

Since only the charge  $q$  and the surface charge  $\sigma$  are inside the surface  $S$ , the total surface charge is equal to  $-q$ . It is perhaps useful to actually do the integral over the surface as a check that we have gotten the charge density there right:

$$\oint_S \sigma da = -\frac{q(1 - \delta^2)}{2} \int_{-1}^1 \frac{dx}{(1 + \delta^2 - 2\delta x)^{3/2}} = -\frac{q(1 - \delta^2)}{2\delta} \frac{1}{(1 + \delta^2 - 2\delta x)^{1/2}} \Big|_{-1}^1 = \quad (2.54)$$

$$= -\frac{q(1 - \delta^2)}{2\delta} \left[ \frac{1}{(1 + \delta^2 - 2\delta)^{1/2}} - \frac{1}{(1 + \delta^2 + 2\delta)^{1/2}} \right] = -\frac{q(1 - \delta^2)}{2\delta} \left[ \frac{1}{(1 - \delta)} - \frac{1}{(1 + \delta)} \right] = -q$$

The total force on the charge due to induced charges on the surface can also be computed. It is equal to force between the real charge and the image charge. In the case of spherical cavity the distance between the charges is  $r'_0 - r_0 = a^2/r_0 - r_0 = a(1/\delta - \delta) = a(1 - \delta^2)/\delta$  and the product of charges is  $q'q = -q^2/\delta$ . Therefore, we obtain

$$|F| = \frac{q^2}{4\pi\epsilon_0 a^2} \frac{\delta}{(1 - \delta^2)^2}. \quad (2.55)$$

The solution of the "inverse" problem which is a point charge outside of a conducting sphere is the same, with the roles of the real charge and the image charge reversed. The only change necessary is in the surface charge density (2.48), where the normal derivative is now outward, implying a change of sign.

### Conducting Sphere in a Uniform Applied Field

Consider next the example of a grounded conducting sphere, which means that  $\Phi(\mathbf{r})=0$  on the sphere, placed in a region of space where there was initially a uniform electric field  $\mathbf{E} = E\hat{\mathbf{z}}$ . Here,  $\hat{\mathbf{z}}$  is a unit vector pointing in the  $z$ -direction. We approach this problem by replacing it with another one which will become equivalent to the first one in some limit. Let us consider the sphere centered at the origin and two charges  $Q$  and  $-Q$  placed at the point  $(0, 0, -d)$  and  $(0, 0, d)$  respectively, as is shown in Fig.2.2.

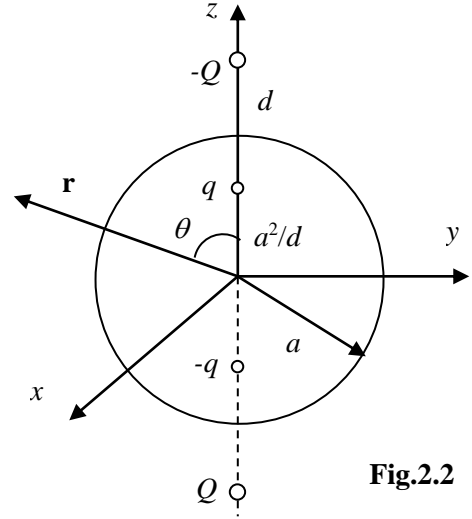


Fig.2.2

The resulting potential configuration can be easily solved by the method of images. There are images of the charges  $\pm Q$  in the sphere at  $(0, 0, -a^2/d)$  and at  $(0, 0, a^2/d)$ . The values of these charges are  $-q = -Qa/d$  and  $q = Qa/d$ , respectively. The potential produced by these charges is

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0} \left[ \frac{1}{|\mathbf{r} + d\hat{\mathbf{z}}|} - \frac{a/d}{|\mathbf{r} + (a^2/d)\hat{\mathbf{z}}|} - \frac{1}{|\mathbf{r} - d\hat{\mathbf{z}}|} + \frac{a/d}{|\mathbf{r} - (a^2/d)\hat{\mathbf{z}}|} \right]. \quad (2.56)$$

It is easy to see that the potential produced by these four charges is zero on the surface of the sphere. Thus we have solved the problem of a grounded sphere in the presence of two symmetrically located equal and opposite charges.

Now we assume that the real charges  $Q$  become increasingly large and move them to infinity in such a way that the field they produce at the origin is constant. Since this field is  $\mathbf{E}(\mathbf{r}) = \frac{2Q}{4\pi\epsilon_0 d^2} \hat{\mathbf{z}}$ , so if  $Q$  is increased at a rate proportional to  $d^2$ , the field at the origin is unaffected. As  $d$  becomes very large in comparison with the radius  $a$  of the sphere, the electric field becomes homogeneous and equal to this value everywhere in the vicinity of the sphere. Hence, we recover the configuration presented in the original problem of a sphere placed in a uniform applied electric field. Hence, if we assume that the electric field in the original problem is given by  $E = \frac{Q}{2\pi\epsilon_0 d^2}$ , or equivalently  $Q = 2\pi\epsilon_0 d^2 E$ , we find the solution from eq.(2.56) by taking the limit of  $d \rightarrow \infty$ :

$$\Phi(\mathbf{r}) = \lim_{d \rightarrow \infty} \frac{d^2 E}{2} \left[ \frac{1}{(r^2 + d^2 + 2rd \cos \theta)^{1/2}} - \frac{\frac{a}{d}}{\left(r^2 + \frac{a^4}{d^2} + 2r \frac{a^2}{d} \cos \theta\right)^{1/2}} \right] +$$

$$\lim_{d \rightarrow \infty} \frac{d^2 E}{2} \left[ -\frac{1}{(r^2 + d^2 - 2rd \cos \theta)^{1/2}} + \frac{\frac{a}{d}}{\left(r^2 + \frac{a^4}{d^2} - 2r \frac{a^2}{d} \cos \theta\right)^{1/2}} \right] =$$

$$\begin{aligned}
&= \lim_{d \rightarrow \infty} \frac{Ed}{2} \left[ \frac{1}{\left(1 + 2\frac{r}{d} \cos \theta + \frac{r^2}{d^2}\right)^{1/2}} - \frac{1}{\left(1 - 2\frac{r}{d} \cos \theta + \frac{r^2}{d^2}\right)^{1/2}} \right] + \\
&\lim_{d \rightarrow \infty} \frac{Ed}{2} \left[ -\frac{\frac{a}{r}}{\left(1 + 2\frac{a^2}{rd} \cos \theta + \frac{a^4}{d^2 r^2}\right)^{1/2}} + \frac{\frac{a}{r}}{\left(1 - 2\frac{a^2}{rd} \cos \theta + \frac{a^4}{d^2 r^2}\right)^{1/2}} \right] = \\
&= \frac{Ed}{2} \left[ -2\frac{r}{d} \cos \theta + \frac{a}{r} 2\frac{a^2}{rd} \cos \theta \right] = -Er \cos \theta + E \frac{a^3}{r^2} \cos \theta
\end{aligned} \tag{2.57}$$

The first term ( $-Ez$ ) is the potential of the applied constant electric field. The second is the potential produced by the induced surface charge density on the sphere or, equivalently, the image charges. This potential has the characteristic form of an electric dipole field. Indeed, the dipole moment  $\mathbf{p}$  is defined as follows

$$\mathbf{p} = \int \rho(\mathbf{r}) \mathbf{r} d^3 r . \tag{2.58}$$

In the present case the dipole moment of the sphere may be found from the image charge distribution. Thus, we find

$$\mathbf{p} = \int \left[ \frac{Qa}{d} \delta(\mathbf{r} - \frac{a^2}{d} \hat{\mathbf{z}}) - \frac{Qa}{d} \delta(\mathbf{r} + \frac{a^2}{d} \hat{\mathbf{z}}) \right] \mathbf{r} d^3 r = \frac{2Qa}{d} \frac{a^2}{d} \hat{\mathbf{z}} = 2\pi\epsilon_0 d^2 E \frac{2a^3}{d^2} \hat{\mathbf{z}} = 4\pi\epsilon_0 E a^3 \hat{\mathbf{z}} . \tag{2.59}$$

Comparison with the expression for the potential shows that the dipolar part of the potential may be written as the standard expression

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} . \tag{2.60}$$

The charge density on the surface of the sphere may be found in the usual way:

$$\sigma = \epsilon_0 E_r = -\epsilon_0 \left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = \epsilon_0 E \cos \theta + 2\epsilon_0 E \frac{a^3}{r^3} \cos \theta \Big|_{r=a} = 3\epsilon_0 E \cos \theta . \tag{2.61}$$

## Orthogonal Functions and Expansions

Now we consider a quite different, much more systematic approach to the solution of Laplace's equation

$$\nabla^2 \Phi = 0 . \tag{2.62}$$

as a boundary value problem. It is implemented by expanding the solution in some domain  $V$  using complete sets of orthogonal functions

$$\Phi(x, y, z) = \sum_{nlm} A_{nlm} U_n(x) V_l(y) W_m(z) . \tag{2.63}$$

and determining the coefficients in the expansion by requiring that the solution takes on the proper values on the boundaries. For simple geometries such as spheres, cylinders, and rectangular parallelepipeds this method can always be utilized. A particular set of orthogonal functions depends on geometry of the

problem. Before launching into a description of how one proceeds in specific geometries, let us to review the terminology of orthogonal function expansions and some basic facts.

Suppose that we have a set of functions  $U_n(x)$ , where  $n = 1, 2, \dots$  which are *orthogonal* on the interval  $(a, b)$ . The orthogonality condition implies that

$$\int_a^b U_n(x)U_m^*(x)dx = 0, \text{ if } n \neq m. \quad (2.64)$$

the superscript \* denotes complex conjugation. If  $n = m$  the integral (2.64) is non-zero. We assume that this integral is equal to unity:

$$\int_a^b U_n(x)U_n^*(x)dx = \int_a^b |U_n(x)|^2 dx = 1. \quad (2.65)$$

Combining these equations we have:

$$\int_a^b U_n(x)U_m^*(x)dx = \delta_{nm}. \quad (2.66)$$

where  $\delta_{nm}$  is the *Kronecker* symbol. The functions  $U_n(x)$  are said to be *orthonormal*.

An arbitrary function  $f(x)$  can be expanded in series of the orthonormal functions  $U_n(x)$  on the interval  $(a, b)$ . Therefore, we attempt to represent an arbitrary function  $f(x)$  as a linear combination of the functions  $U_n(x)$ , which are referred to as basis functions. Keeping just  $N$  terms in the expansion, we have

$$f(x) \approx \sum_{n=1}^N A_n U_n(x). \quad (2.67)$$

We need a criterion for choosing the coefficients  $A_n$  in the expansion. A standard criterion is to minimize the mean square error  $E$  which is defined as follows:

$$E = \int_a^b \left| f(x) - \sum_{n=1}^N A_n U_n(x) \right|^2 dx = \int_a^b \left[ f(x) - \sum_{n=1}^N A_n U_n(x) \right] \left[ f^*(x) - \sum_{m=1}^N A_m^* U_m^*(x) \right] dx. \quad (2.68)$$

The conditions for an extremum are

$$\frac{\partial E}{\partial A_k} = 0; \quad \frac{\partial E}{\partial A_k^*} = 0, \quad (2.69)$$

where  $A_k$  and  $A_k^*$  have been treated as independent variables (this is always possible because they are linearly related to  $\text{Re}(A_k)$  and  $\text{Im}(A_k)$ ). Application of the second of these conditions leads to

$$0 = \frac{\partial E}{\partial A_k^*} = \int_a^b \left[ f(x) - \sum_{n=1}^N A_n U_n(x) \right] U_k^*(x) dx. \quad (2.70)$$

Using the orthonormality of the basis functions, we find

$$A_k = \int_a^b f(x) U_k^*(x) dx. \quad (2.71)$$

Similar we can find that  $A_k^*$  is given by the complex conjugate of this relation.

If the number of terms  $N$  in expansion (2.67) becomes larger and larger, we intuitively expect that our series representation of the function  $f(x)$  becomes “better” and “better”. This is indeed correct if the set of basis functions  $U_n(x)$  is *complete*. This means that the mean square error can be made arbitrarily small by keeping a sufficiently large number of terms in the sum. Then one says that the sum converges in the mean to the given function.

$$f(x) = \sum_{n=1}^{\infty} A_n U_n(x). \quad (2.72)$$

Series (2.72) can be written with the explicit form (2.71) for the coefficients which leads to

$$f(x) = \sum_{n=1}^{\infty} U_n(x) \int_a^b f(x') U_n^*(x') dx' = \int_a^b \left[ \sum_{n=1}^{\infty} U_n(x) U_n^*(x') \right] f(x') dx'. \quad (2.73)$$

It is evident from eq.(2.73) that

$$\sum_{n=1}^{\infty} U_n(x) U_n^*(x') = \delta(x - x'). \quad (2.74)$$

This equation is called the *completeness* or *closure relation*.

This description can be easily generalized to a space of arbitrary dimension. For example, in two dimensions we may have the space of  $x$  and  $y$  with defined at intervals  $(a,b)$  and  $(c,d)$  respectively and complete sets of orthonormal functions  $U_n(x)$  and  $V_m(y)$  on the respective intervals. Then the arbitrary function  $f(x,y)$  can be expanded as follows

$$f(x, y) = \sum_{n,m=1}^{\infty} A_{nm} U_n(x) V_m(y), \quad (2.75)$$

where

$$A_{nm} = \int_a^b dx \int_c^d dy f(x, y) U_n^*(x) V_m^*(y). \quad (2.76)$$

## Fourier Series

An important case of orthogonal functions are the sines and cosines. An expansion in terms of these functions is a *Fourier series*. Assume that we have the interval  $-a/2 < x < a/2$ . Then the Fourier series may be built from the basis functions

$$\sqrt{\frac{2}{a}} \sin\left(\frac{2\pi nx}{a}\right), \quad \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi nx}{a}\right). \quad (2.77)$$

where  $n$  is a non-negative integer and for  $n = 0$  the cosine function is replaced by  $\sqrt{1/a}$ . The series which is equivalent to (2.72) is customary written in the form

$$f(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left[ A_n \cos\left(\frac{2\pi nx}{a}\right) + B_n \sin\left(\frac{2\pi nx}{a}\right) \right], \quad (2.78)$$

where

$$A_n = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \cos\left(\frac{2\pi nx}{a}\right) dx$$

$$B_n = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \sin\left(\frac{2\pi nx}{a}\right) dx. \quad (2.79)$$

If the interval  $(-a/2, +a/2)$  becomes infinite,  $-\infty < x < \infty$ , then the index  $n$  of the functions  $U_n(x)$  becomes a continuous index, and the Kronecker symbol in eq.(2.66) becomes a Dirac delta function. This brings us to the *Fourier integral* which is the limit of a Fourier series when the interval on which functions are expanded becomes infinite. Starting from the orthonormal set of complex exponentials

$$U_n(x) = \frac{1}{\sqrt{a}} e^{i(2\pi nx/a)}, \quad (2.80)$$

where  $n = 0, \pm 1, \pm 2, \dots$  on the interval  $-a/2 < x < a/2$ , the Fourier series has a form

$$f(x) = \frac{1}{\sqrt{a}} \sum_{n=-\infty}^{\infty} A_n e^{i(2\pi nx/a)}, \quad (2.81)$$

where

$$A_n = \frac{1}{\sqrt{a}} \int_{-a/2}^{a/2} f(x) e^{-i(2\pi nx/a)} dx. \quad (2.82)$$

Then let the interval become infinite ( $a \rightarrow \infty$ ), at the same time transforming

$$\begin{aligned} \frac{2\pi n}{a} &\rightarrow k \\ \sum_n &\rightarrow \int_{-\infty}^{\infty} dn = \frac{a}{2\pi} \int_{-\infty}^{\infty} dk. \\ A_n &\rightarrow \sqrt{\frac{2\pi}{a}} A(k) \end{aligned} \quad (2.83)$$

The resulting expansion, equivalent to (2.81), is the *Fourier integral*,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk, \quad (2.84)$$

where

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \quad (2.85)$$

The orthogonality condition is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x} dx = \delta(k-k'), \quad (2.86)$$

while the completeness relation is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk = \delta(x-x'). \quad (2.87)$$

This shows a complete equivalentness of the two continuous variables  $x$  and  $k$ .

### Laplace Equation in Rectangular Coordinates

Now we consider the approach to the solution of the Laplace equation in rectangular coordinates

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0. \quad (2.88)$$

A solution of this partial differential equation can be found in terms of three ordinary differential equations, by the assumption that the potential can be represented by a product of three functions, one for each coordinate

$$\Phi(x, y, z) = X(x)Y(y)Z(z). \quad (2.89)$$

Substituting this into the Laplace equation yields

$$Y(y)Z(z)\frac{\partial^2 X}{\partial x^2} + X(x)Z(z)\frac{\partial^2 Y}{\partial y^2} + X(x)Y(y)\frac{\partial^2 Z}{\partial z^2} = 0. \quad (2.90)$$

Dividing by  $X(x)Y(y)Z(z)$  results in

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0. \quad (2.91)$$

Each term in this equation depends on a single variable. Consequently, since the equation must remain valid when any of the variables is changed while the others held fixed, it must be the case that each term is a constant, independent of the variable. Since the three terms add to zero, at least one must be a positive constant, and at least one must be a negative constant. Let us suppose that two are negative, and one, positive. Thus we have

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\alpha^2; \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -\beta^2; \quad \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \gamma^2, \quad (2.92)$$

where

$$\gamma^2 = \alpha^2 + \beta^2. \quad (2.93)$$

For negative constants, the solutions are oscillatory. For positive constant the solutions are exponential:

$$X(x) \sim e^{\pm i\alpha x}; \quad Y(y) \sim e^{\pm i\beta y}; \quad Z(z) \sim e^{\pm \gamma z}, \quad (2.94)$$

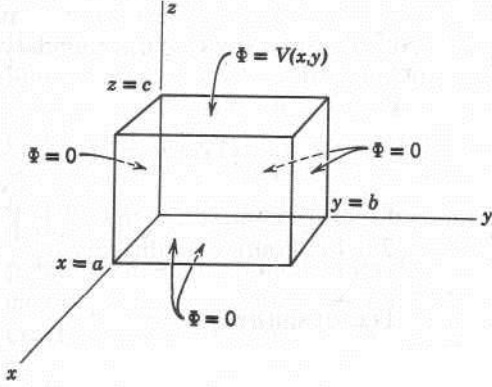
or, equivalently,

$$\begin{aligned} X(x) &\sim \sin(\alpha x) \text{ or } \cos(\alpha x), \\ Y(y) &\sim \sin(\beta y) \text{ or } \cos(\beta y), \\ Z(z) &\sim \sinh(\gamma z) \text{ or } \cosh(\gamma z). \end{aligned} \quad (2.95)$$

At this stage  $\alpha$  and  $\beta$  are arbitrary real constants, so that by taking linear combinations of solutions of the kind outlined above, we can construct any function of  $x$  and  $y$  at some particular value of  $z$ .

This is just what we need to solve boundary value problems with planar surfaces. For example, suppose that we need to solve the Laplace equation inside of a rectangular parallelepiped of edge lengths  $a$ ,  $b$ , and  $c$  with the potential given on the surface. We can find a solution by considering six distinct problems and superposing the six solutions to them. Each of these six problems has a given potential on one face of the box (given in the original problem) while on the other five faces the potential is zero. Summing the six solutions gives a potential which has the same values on each face of the box as given in the original problem. Let's see how to solve one of these six problems.

Suppose that the faces of the box are given by the planes  $z = 0$ ,  $c$ ;  $x = 0$ ,  $a$ ; and  $y = 0$ ,  $b$  (Fig.2.3). Let the potential on the face  $z = c$  be  $\Phi(x, y, c) = V(x, y)$ , while  $\Phi(x, y, z) = 0$  on the other five faces. It is required to find the potential everywhere inside the box.



**Fig.2.3** Hollow, rectangular box with five sides at zero potential, while the sixth ( $z = c$ ) has the specified potential  $\Phi(x, y, c) = V(x, y)$ .

We need to satisfy the boundary conditions. Starting from the requirement that  $\Phi = 0$  for  $x = 0, y = 0, z = 0$ , it is easy to see that the required forms of  $X, Y$ , and  $Z$  are

$$\begin{aligned} X(x) &= \sin(\alpha x), \\ Y(y) &= \sin(\beta y), \\ Z(z) &= \sinh(\gamma z). \end{aligned} \quad (2.96)$$

To have  $\Phi = 0$  for  $x = a$  and  $y = b$  we must have  $\alpha = \pi n / a$  and  $\beta = \pi m / b$ . With the definitions

$$\begin{aligned} \alpha_n &= \pi n / a, \\ \beta_m &= \pi m / b, \\ \gamma_{nm} &= \pi \sqrt{n^2 / a^2 + m^2 / b^2}, \end{aligned} \quad (2.97)$$

we can write the partial potential, satisfying all the boundary conditions except one,

$$\Phi_{nm}(x, y, z) = \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z). \quad (2.98)$$

The potential can be expanded in terms of these  $\Phi_{nm}$  with coefficients to be chosen to satisfy the final boundary condition:

$$\Phi(x, y, z) = \sum_{n,m=1}^{\infty} A_{nm} \Phi_{nm} = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z). \quad (2.99)$$

There remain only the boundary condition  $\Phi = V(x, y)$  at  $z = c$ :

$$V(x, y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c). \quad (2.100)$$

This is a double Fourier series expansion for function  $V(x, y)$ . Consequently the coefficients are given by

$$A_{nm} = \frac{4}{ab \sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy V(x, y) \sin(\alpha_n x) \sin(\beta_m y). \quad (2.101)$$

For example, if the potential  $V(x, y) = V_0$  is constant then the coefficients are given by

$$A_{nm} = \frac{16V_0}{\pi^2 nm \sinh(\gamma_{nm} c)} \begin{cases} 0, & \text{if } n \text{ or } m \text{ are even;} \\ 1, & \text{if } n \text{ and } m \text{ are odd.} \end{cases} \quad (2.102)$$



The solution for the potential in this case takes the form:

$$\Phi(x, y, z) = \frac{16V_0}{\pi^2} \sum_{n,m=0}^{\infty} \frac{1}{(2n+1)(2m+1)\sinh(\gamma_{nm}c)} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z), \quad (2.103)$$

where

$$\begin{aligned} \alpha_n &= \pi(2n+1)/a, \\ \beta_m &= \pi(2m+1)/b, \\ \gamma_{nm} &= \pi\sqrt{(2n+1)^2/a^2 + (2m+1)^2/b^2}. \end{aligned} \quad (2.104)$$

As we have already explained, if the rectangular box has potentials different from zero on all six sides, the required solution for the potential inside the box can be obtained by a linear superposition of six solutions, one for each side, equivalent to (2.99) and (2.101). The problem of the solution of the Poisson equation, that is, the potential inside the box with a charge distribution inside, as well as prescribed boundary conditions on the surface, requires the construction of the appropriate Green function.