

Section 1: Electrostatics

Charge density

The most fundamental quantity of electrostatics is electric charge.

Charge comes in two varieties, which are called “positive” and “negative”, because their effects tend to cancel. For example if we have $+q$ and $-q$ at the same point, electrically it is the same as no charge at all.

Charge is conserved: it cannot be created or destroyed. The total charge of a closed system does not change. This is called global conservation of charge. It may happen, however, that charge disappear in one point in space and instantly appear in the other point due to the motion of the charge. There should be then continuous path between the points. This is called local conservation of charge which has its manifestation in the continuity equation.

Charge is quantized. The fact is that electric charges come only in discrete lumps – integers multiples of the basic unit charge e . For example electron carries charge $-e$ and proton carries charge $+e$. The fundamental unit of charge is so tiny that in macroscopic applications this charge quantization can be ignored. We will therefore be dealing with continuous distributions of charge treating it as a continuous fluid.

The fundamental quantity which describes this charge distribution is the charge density. The charge density is the local charge per unit volume, i.e.

$$\rho(\mathbf{r}) = \lim_{\Delta V \rightarrow 0} \Delta q / \Delta V = dq / dV , \quad (1.1)$$

where ΔV is the volume centered at point \mathbf{r} .

The total charge in volume V is therefore

$$Q = \int_V dq = \int_V \rho(\mathbf{r}) dV = \int_V \rho(\mathbf{r}) d^3r , \quad (1.2)$$

Since the SI unit of charge is Coulomb, the unit of charge density is C/m^3 .

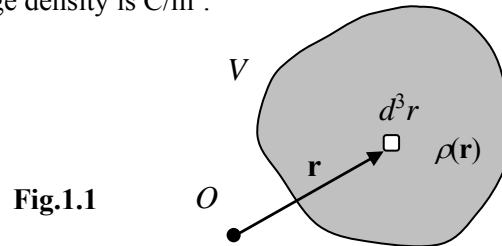


Fig.1.1

If charge Q is localized in a point of space, say \mathbf{r}_0 , this charge is called a *point charge*. A point charge produces an infinite charge density at the point \mathbf{r}_0 . Indeed, if we take volume ΔV centered at point \mathbf{r}_0 and tend this volume to zero, the ratio $\Delta q / \Delta V$ will diverge due to the constant charge $\Delta q = Q$ located in this volume independent of the magnitude of the volume. Therefore, the charge density $\rho(\mathbf{r}_0) = \infty$. At any other points $\mathbf{r} \neq \mathbf{r}_0$ the charge density is zero, $\rho(\mathbf{r}) = 0$. Despite the divergence of the charge density at $\mathbf{r} = \mathbf{r}_0$ the integral over any volume V which contains point $\mathbf{r} = \mathbf{r}_0$ is finite and is equal to the total charge in this volume: $\int_V \rho(\mathbf{r}) dV = Q$. In this case the charge density is given in terms of the *delta function*:

$$\rho(\mathbf{r}) = Q \delta^3(\mathbf{r} - \mathbf{r}_0) . \quad (1.3)$$

In one dimension, the delta function, written $\delta(x - a)$, is a mathematically improper function having the

properties:

$$\delta(x-a) = 0, \text{ if } x \neq a \quad (1.4)$$

and

$$\int \delta(x-a) dx = 1. \quad (1.5)$$

The delta function can be given an intuitive meaning as the limit of a peaked curve such as a Lorentzian that becomes narrower and narrower, but higher and higher, in such a way that the area under the curve is always constant.

$$\delta(x) = \lim_{\Gamma \rightarrow 0} \frac{\Gamma}{2\pi} \frac{1}{x^2 + \Gamma^2/4}. \quad (1.6)$$

From the definitions above it is evident that, for an arbitrary function $f(x)$,

$$\int f(x) \delta(x-a) dx = f(a). \quad (1.7)$$

In three dimensions, with Cartesian coordinates,

$$\delta^3(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0) \delta(y - y_0) \delta(z - z_0) \quad (1.8)$$

is a function that vanishes everywhere except at $\mathbf{r} = \mathbf{r}_0$, and is such that

$$\int_V \delta^3(\mathbf{r} - \mathbf{r}_0) d^3r = \begin{cases} 1, & \text{if } \mathbf{r}_0 \in V \\ 0, & \text{if } \mathbf{r}_0 \notin V \end{cases} \quad (1.9)$$

Note that a delta function has the dimensions of an inverse volume.

For N point charges q_i located at positions \mathbf{r}_i ($i = 1 \dots N$) the charge density is given by

$$\rho(\mathbf{r}) = \sum_{i=1}^N q_i \delta^3(\mathbf{r} - \mathbf{r}_i). \quad (1.10)$$

In addition to volume charge density we will also be dealing with surface and line charge densities. The surface charge density $\sigma(\mathbf{r})$ is given the local surface charge per unit surface, such that the charge in a small surface area da at point \mathbf{r} is then $dq = \sigma(\mathbf{r}) da$. Similarly, the line charge density $\lambda(\mathbf{r})$ is given by the local line charge per unit length dl , such that the charge in a small element of length dl at point \mathbf{r} is then $dq = \lambda(\mathbf{r}) dl$.

Coulomb's law

All of electrostatics originates from the quantitative statement of Coulomb's law concerning the force acting between charged bodies at rest with respect to each other. Coulomb, in a series of experiments, showed experimentally that the force between two small charged bodies separated in air a distance large compared to their dimensions

- varies directly as the magnitude of each charge,
- varies inversely as the square of the distance between them,
- is directed along the line joining the charges, and
- is attractive if the bodies are oppositely charged and repulsive if the bodies have the same type of charge.

If the two charges located at positions \mathbf{r}_1 and \mathbf{r}_2 the force on the charge at \mathbf{r}_2 is equal to

$$\mathbf{F} = \frac{q_1 q_2 (\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\epsilon_0 |\mathbf{r}_2 - \mathbf{r}_1|^3}. \quad (1.11)$$

Here we use the SI system, in which ϵ_0 is the *permittivity of free space* $\epsilon_0 = 8.85 \cdot 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$.*

Experiments show that the electrostatic force is additive so that the force on a point charge q placed at position \mathbf{r} due to other charges q_i located at positions \mathbf{r}_i ($i = 1 \dots N$) is given by

$$\mathbf{F} = \frac{q}{4\pi\epsilon_0} \sum_{i=1}^N q_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3} \quad (1.12)$$

This relation is called the *superposition principle*.

Although the thing that eventually gets measured is a force, it is useful to introduce a concept of an electric field. The electric field is defined as the force exerted on a unit charge located at position vector \mathbf{r} . It is a vector function of position, denoted by \mathbf{E} .

$$\mathbf{F} = q\mathbf{E}, \quad (1.13)$$

where \mathbf{F} is the force, \mathbf{E} the electric field, and q the charge. According to Coulomb the electric field at the point \mathbf{r} due to point charges q_i located at positions \mathbf{r}_i ($i=1 \dots N$) is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^N q_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}, \quad (1.14)$$

which is superposition of field produced by the individual point charges. In the SI system, the electric field is measured in *volts per meter* (V/m).

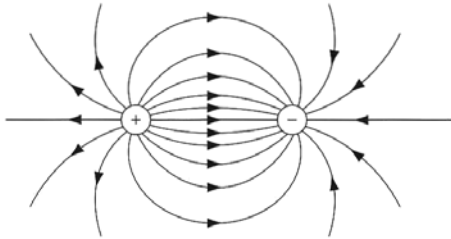


Fig.1.2

The electric field can be visualized using field lines as is shown schematically in Fig.1.2 for two charges, positive and negative. Positive charges can be thought as a source of an electric field and negative charges can be thought as sinks of an electric field. The magnitude of the electric field is determined by the density of the field lines.

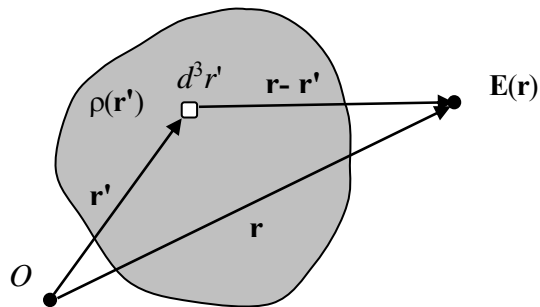


Fig.1.3

* In SI, the permittivity is defined using $\epsilon_0\mu_0 = 1/c^2$, where μ_0 is the *permeability of free space*. This constant, relevant for magnetic phenomena, will be discussed in Sec. 5. In SI it is defined to have the exact value $\epsilon_0\mu_0 = 4\pi \cdot 10^{-7} \text{ N} / \text{A}^2$.

If there is continuous distribution of the charge it can be described by a charge density $\rho(\mathbf{r}')$ as is shown schematically in Fig.1.3. In this case the sum in Eq.(1.14) is replaced by an integral:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V dq \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3r' . \quad (1.15)$$

Similarly we can write expressions for the field produced by surface and line charge densities. For the surface charge density we find

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_S \sigma(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} da' , \quad (1.16)$$

where the integral is taken over surface S . For the line charge density we have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_C \lambda(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dl' , \quad (1.17)$$

where the integral is taken over line C .

Gauss's law

The integral above is not always the most suitable form for the evaluation of electric fields. There is an important integral expression, called *Gauss's law*, which is sometimes more useful and furthermore leads to a differential equation for $\mathbf{E}(\mathbf{r})$.

First, we define the flux of electric field through a surface S by the integral of the normal component of the electric field over the surface, $\int_S \mathbf{E} \cdot \mathbf{n} da$, where \mathbf{n} is the normal to this surface. The flux is the measure of the “number of field lines” passing through S . This quantity depends on the field strength and therefore controlled by the charge producing the electric field.

The flux over any *closed* surface is a measure of the total charge inside. Indeed for charges inside the closed surface the field lines starting on positive charges should terminate on negative charges inside or pass through the surface. On the other hand, charges outside the surface will contribute nothing to the total flux, since its field lines pass in one side and out the other. This is the essence of Gauss's law. Now we will make it quantitative.

Consider a point charge q and a *closed* surface S , as shown in Fig. 1.4.

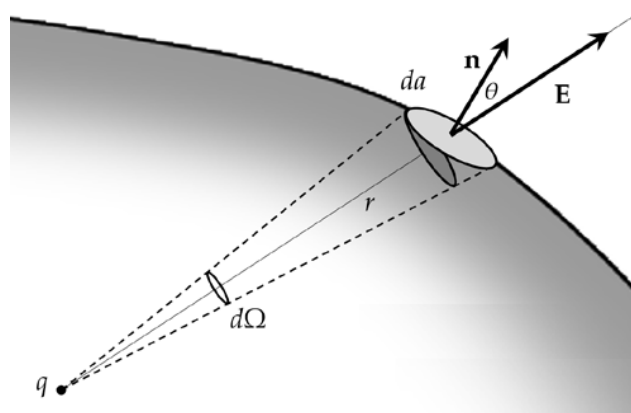


Fig.1.4

Let r be the distance from the charge to a point on the surface, \mathbf{n} be the outwardly directed unit normal to the surface at that point, da be an element of surface area. If the electric field \mathbf{E} at the point on the surface due to the charge q makes an angle θ with the unit normal, then the normal component of \mathbf{E} times the area element is:

$$\mathbf{E} \cdot \mathbf{n} da = \frac{q}{4\pi\epsilon_0} \frac{\cos\theta}{r^2} da . \quad (1.18)$$

Since \mathbf{E} is directed along the line from the charge q to the surface element, $\cos\theta da = r^2 d\Omega$, where $d\Omega$ is the element of solid angle subtended by da at the position of the charge. Therefore

$$\mathbf{E} \cdot \mathbf{n} da = \frac{q}{4\pi\epsilon_0} d\Omega . \quad (1.19)$$

If we now integrate the normal component of \mathbf{E} over the whole surface, we find that

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = \begin{cases} q/\epsilon_0, & \text{if } q \text{ is inside } S \\ 0, & \text{if } q \text{ is outside } S \end{cases} \quad (1.20)$$

The result for the charge outside the volume follows from the fact that the incoming flux of the electric field

$$\mathbf{E}_1 \cdot \mathbf{n}_1 da = \frac{q}{4\pi\epsilon_0} \frac{\cos\theta_1}{r_1^2} da_1 = -\frac{q}{4\pi\epsilon_0} d\Omega \quad (1.21)$$

is canceled out by the outgoing flux of the electric field through the opposite surface

$$\mathbf{E}_2 \cdot \mathbf{n}_2 da = \frac{q}{4\pi\epsilon_0} \frac{\cos\theta_2}{r_2^2} da_2 = \frac{q}{4\pi\epsilon_0} d\Omega , \quad (1.22)$$

as is seen from Fig.1.5.

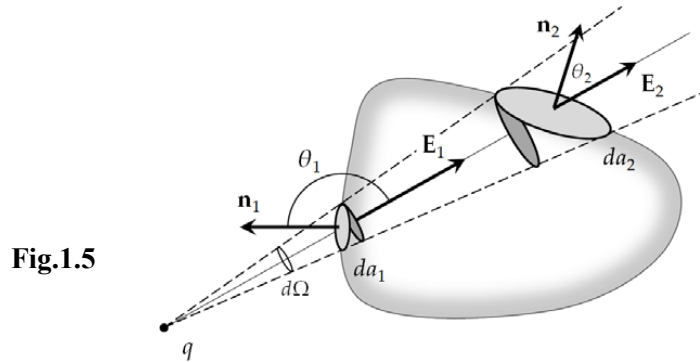


Fig.1.5

Notice that in eq.(1.20) the distance to the surface is cancelled out. This is because the surface area goes up as r^2 , the field goes down as $1/r^2$, so that the flux remains constant. Evidently the flux through any surface enclosing the charge is q/ϵ_0 .

This result is Gauss's law for a single point charge. For a discrete set of charges, due to the superposition principle it is immediately apparent that

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = \frac{1}{\epsilon_0} \sum_i q_i , \quad (1.23)$$

where the sum is over only those charges *inside* the surface S . For a continuous charge density $\rho(\mathbf{r})$,

Gauss's law becomes:

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{r}) d^3 r = \frac{1}{\epsilon_0} Q, \quad (1.24)$$

where V is the volume enclosed by S and Q is the total charge in the volume V . This equation is one of the basic equations of electrostatics. Note that it depends upon the inverse square law for the force between charges, the central nature of the force, and the linear superposition of the effects of different charges. Therefore, Gauss's law is the consequence of Coulomb's law.

Applications of Gauss's Law

Gauss's law provides a powerful tool for determining the electric field where the problem has spherical or cylindrical symmetry.

Spherical Symmetry

Suppose we have a spherically symmetric distribution of charge $\rho(\mathbf{r}) = \rho(r)$, where $r = |\mathbf{r}|$. By symmetry, the electric field depends only on r , and therefore must be in the radial direction $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$. Choose a spherical surface S of radius r , centered on the center of the charge distribution. Then we have

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = \oint_S E(r)\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} da = E(r)4\pi r^2, \quad (1.25)$$

But by Gauss's law, we have

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{r}') d^3 r' = \frac{Q(r)}{\epsilon_0}, \quad (1.26)$$

where $Q(r)$ is the total charge contained within the sphere of radius r . Thus we obtain

$$\mathbf{E}(\mathbf{r}) = \frac{Q(r)}{4\pi r^2 \epsilon_0} \hat{\mathbf{r}}, \quad (1.27)$$

Note that outside a spherically symmetric charge distribution, the field is the same as if we had a point-like charge $Q(r)$ at the origin.

For a thin spherical shell of charge Q , we can immediately conclude that

(i) Outside the shell, the electric field is the same as that of the equivalent point charge Q at its center:

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi r^2 \epsilon_0} \hat{\mathbf{r}}, \quad (1.28)$$

(ii) Inside the shell, the field is zero.

Cylindrical Symmetry

Suppose we have a infinitely long, cylindrically symmetric distribution of charge, with the axis of symmetry along the z axis. We introduce cylindrical coordinates (s, φ, z) and consider an element of length L , and radius s , containing a charge $Q(s, L)$. The field depends solely on s , and therefore must be in the $\hat{\mathbf{s}}$ direction, $\mathbf{E}(\mathbf{r}) = E(s)\hat{\mathbf{s}}$. Applying Gauss's law to the cylinder we have

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = \frac{Q(s, L)}{\epsilon_0}. \quad (1.29)$$

Since there is no flux across the flat surfaces of the cylinder at $z = 0$ and $z = L$, we obtain

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = E(s) \oint_S da = E(s) 2\pi s L . \quad (1.30)$$

Thus

$$\mathbf{E}(s) = \frac{Q(s, L)}{2\pi\epsilon_0 s L} \hat{\mathbf{s}} . \quad (1.31)$$

For example, in case of infinitely long, thin rod carrying charge λ per unit length, $Q(s, L) = \lambda L$ and we have

$$\mathbf{E}(s) = \frac{\lambda}{2\pi\epsilon_0 s} \hat{\mathbf{s}} . \quad (1.32)$$

Differential form of Gauss's Law

Gauss's law can be thought of as being an integral formulation of the law of electrostatics. We can obtain a differential form (i.e., a differential equation) by using the divergence theorem. The *divergence theorem* states that for any well-behaved vector field $\mathbf{A}(\mathbf{r})$ defined within a volume V surrounded by the closed surface S the relation

$$\oint_S \mathbf{A} \cdot \mathbf{n} da = \int_V \nabla \cdot \mathbf{A} d^3 r \quad (1.33)$$

holds between the volume integral of the divergence of \mathbf{A} and the surface integral of the outwardly directed normal component of \mathbf{A} . The equation in fact can be used as the definition of the divergence.

The divergence theorem allows us to write Eq.(1.24) as

$$\int_V \left(\nabla \cdot \mathbf{E} - \frac{\rho}{\epsilon_0} \right) d^3 r = 0 . \quad (1.34)$$

Since this equation is valid for an arbitrary volume V , we can put the integrand equal to zero to obtain

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} , \quad (1.35)$$

which is the differential form of Gauss's law of electrostatics.

The divergence of \mathbf{E}

Now let us go back and calculate the divergence of \mathbf{E} directly from eq. (1.15) which says that

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r' . \quad (1.36)$$

Taking into account that the differentiation is performed with respect to the unprimed coordinates we have

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \left(\nabla \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) d^3 r' . \quad (1.37)$$

The divergence under the integral can be calculated as follows. We consider $\nabla \cdot \frac{\mathbf{r}}{r^3}$. For $r \neq 0$

$$\nabla \cdot \frac{\mathbf{r}}{r^3} = \mathbf{r} \cdot \nabla \left(\frac{1}{r^3} \right) + \frac{1}{r^3} (\nabla \cdot \mathbf{r}) = -\mathbf{r} \cdot \frac{3\mathbf{r}}{r^5} + \frac{3}{r^3} = 0 . \quad (1.38)$$

For $r = 0$ the divergence is undefined. Applying the divergence theorem for arbitrary volume containing

the origin we find

$$\int \nabla \cdot \frac{\mathbf{r}}{r^3} dV = \oint \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} da = \oint \frac{r}{r^3} \cos \theta da = \oint \frac{r}{r^3} r^2 d\Omega = 4\pi . \quad (1.39)$$

Thus the function $\frac{1}{4\pi} \nabla \cdot \frac{\mathbf{r}}{r^3}$ has the properties of the delta function – it is equal to zero everywhere in space except $r=0$ where it diverges, and the integral over arbitrary volume containing point $r=0$ is equal to unity. Therefore,

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) = 4\pi \delta^3(\mathbf{r}) . \quad (1.40)$$

This results is consistent with Gauss's law $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$ for a point charge at the origin for which

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^3} \quad \text{and} \quad \rho(\mathbf{r}) = q\delta^3(\mathbf{r}) .$$

Replacing the origin to the point \mathbf{r}' we obtain

$$\nabla \cdot \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) = 4\pi \delta^3(\mathbf{r} - \mathbf{r}') . \quad (1.41)$$

Substituting this results to eq.(1.37) we find

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{r}') \delta^3(\mathbf{r} - \mathbf{r}') d^3 r' = \frac{\rho(\mathbf{r})}{\epsilon_0} . \quad (1.42)$$

Scalar Potential

The single equation (1.35) is not enough to specify completely the three components of the electric field $\mathbf{E}(\mathbf{r})$. A vector field can be specified completely if its divergence and curl are given everywhere in space. Thus we look for an equation specifying curl \mathbf{E} as a function of position. Such an equation follows directly from our generalized Coulomb's law (1.15). The vector factor in the integrand is the negative gradient of the scalar $1/|\mathbf{r} - \mathbf{r}'|$:

$$\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = -\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} . \quad (1.43)$$

Then the field can be written

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \nabla \int \rho(\mathbf{r}') \frac{1}{|\mathbf{r} - \mathbf{r}'|} d^3 r' . \quad (1.44)$$

Since the curl of the gradient of any well-behaved scalar function of position vanishes ($\nabla \times \nabla \psi = 0$, for all ψ), it follows immediately from this equation that

$$\nabla \times \mathbf{E} = 0 . \quad (1.45)$$

Eq. (1.45) is the consequence of the central nature of the force between charges and of the fact that the force is a function of relative distances only.

According to Eq.(1.44) the electric field (a vector) is derived from a scalar by the gradient operation. Since one function of position is easier to deal with than three, it is worthwhile to define the *scalar potential* $\Phi(\mathbf{r})$ by the equation:

$$\mathbf{E} = -\nabla\Phi . \quad (1.46)$$

Then the scalar potential is given in terms of the charge density by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' . \quad (1.47)$$

where the integration is over all the space, and Φ is arbitrary only to the extent that a constant can be added to the right-hand side of this equation. It is obvious from this expression that the potential of a point charge q placed at point $\mathbf{r} = \mathbf{r}_0$ so that $\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}_0)$ is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r} - \mathbf{r}_0|} . \quad (1.48)$$

The scalar potential has a physical interpretation when we consider the work done on a test charge q in transporting it from one point A to another point B in the presence of an electric field $\mathbf{E}(\mathbf{r})$, as is shown in Fig. 1.6.

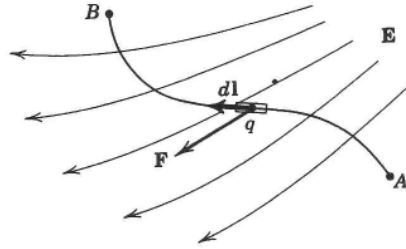


Fig.1.6

The force acting on the charge at any point is $\mathbf{F} = q\mathbf{E}$, so that the work done in moving the charge from A to B is

$$W = -\int_A^B \mathbf{F} \cdot d\mathbf{l} = -q \int_A^B \mathbf{E} \cdot d\mathbf{l} . \quad (1.49)$$

The minus sign appears because we are calculating the work done *on* the charge against the action of the field. The work can be written

$$W = q \int_A^B \nabla\Phi \cdot d\mathbf{l} = q \int_A^B d\Phi = q(\Phi_B - \Phi_A) . \quad (1.50)$$

This shows that $q\Phi$ can be interpreted as the potential energy of the test charge in the electrostatic field.

From (1.49) and (1.50) it can be seen that the line integral of the electric field between two points is independent of the path and is the negative of the potential difference between the points:

$$\int_A^B \mathbf{E} \cdot d\mathbf{l} = -(\Phi_B - \Phi_A) . \quad (1.51)$$

This statement defines a *conservative* vector field. If the path is closed, the line integral is zero,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 . \quad (1.52)$$

a result that can also be obtained directly from Coulomb's law.

Stokes's theorem says that if $\mathbf{A}(\mathbf{r})$ is a well-behaved vector field, S is an arbitrary open surface, and C is the closed curve bounding S , then

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} da . \quad (1.53)$$

where $d\mathbf{l}$ is a line element of C , \mathbf{n} is the normal to S , and the path C is traversed in a right-hand screw sense relative to \mathbf{n} leads immediately back to $\nabla \times \mathbf{E} = 0$.

Poisson and Laplace Equations

We see that the behavior of an electrostatic field can be described by the two differential equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} , \quad (1.54)$$

$$\nabla \times \mathbf{E} = 0 . \quad (1.55)$$

the latter equation being equivalent to the statement that \mathbf{E} is the gradient of a scalar function, the scalar potential Φ :

$$\mathbf{E} = -\nabla\Phi . \quad (1.56)$$

Equations (1.54) and (1.56) can be combined into one partial differential equation for the single function $\Phi(\mathbf{x})$:

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} . \quad (1.57)$$

This equation is called the *Poisson equation*. In regions of space that lack a charge density, the scalar potential satisfies the *Laplace equation*:

$$\nabla^2\Phi = 0 . \quad (1.58)$$

Expression (1.47) we obtained earlier for the potential obeys the *Poisson equation* (1.57). Substituting eq. (1.47) to (1.57) we find

$$\nabla^2\Phi = \frac{1}{4\pi\epsilon_0} \nabla^2 \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') \nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) d^3r' = -\frac{\rho}{\epsilon_0} . \quad (1.59)$$

Since the charge density can be quite general, the quantity in large parentheses above satisfies the condition placed on the delta function:

$$\int f(\mathbf{r}') \delta^3(\mathbf{r}-\mathbf{r}') d^3r' = f(\mathbf{r}) , \quad (1.60)$$

hence we conclude that

$$\nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}') . \quad (1.61)$$

This result is expectable because $\frac{1}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}-\mathbf{r}'|}$ is the potential of a unit point charge placed at \mathbf{r}' , and $\delta^3(\mathbf{r}-\mathbf{r}')$ is the corresponding charge density. Thus, eq. (1.61) presents Poisson equation for a unit point charge located at \mathbf{r}' .

As an exercise we may derive this result in a different way. Using spherical coordinates, we find that

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \frac{1}{r} \right) = 0 \quad (1.62)$$

for $r \neq 0$. At $r = 0$, where $1/r$ diverges, $\nabla^2(1/r)$ is undefined. The integral of $\nabla^2(1/r)$ over an arbitrary volume V containing the origin is, however, finite:

$$\int_V \nabla^2\left(\frac{1}{r}\right) d^3r = \int_V \nabla \cdot \nabla\left(\frac{1}{r}\right) d^3r = \oint \nabla\left(\frac{1}{r}\right) \cdot \mathbf{n} da = -\oint \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} da = -\oint \frac{1}{r^2} \cos\theta da = -\oint \frac{1}{r^2} r^2 d\Omega = -4\pi. \quad (1.63)$$

Thus, we have shown that

$$\nabla^2\left(\frac{1}{r}\right) = 0, \quad r \neq 0; \quad (1.64)$$

$$-\frac{1}{4\pi} \int_V \nabla^2\left(\frac{1}{r}\right) d^3r = 1, \quad r = 0 \in V. \quad (1.65)$$

This result tells us that

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\mathbf{r}). \quad (1.66)$$

Surface charge distribution

One of the common problems of electrostatics is the determination of electric field and the potential due to a given distribution of surface charges. Gauss's law allows us to find the result. Suppose that we have a Gaussian pillbox, extending barely over the edge above and below the surface as is shown in Fig.1.7.

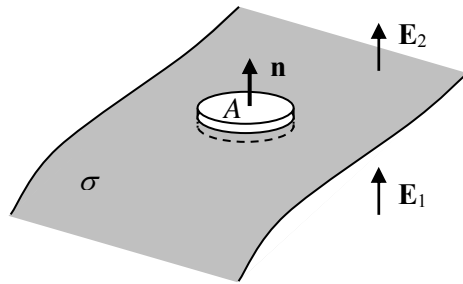


Fig.1.7

Gauss's law states that

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = \frac{\sigma A}{\epsilon_0}, \quad (1.67)$$

where A is the area of the pillbox. Note that if σ is inhomogeneous then A should be sufficiently small. Since the sides of the pillbox contribute nothing to the flux in the limit when the thickness of the pillbox tends to zero, we obtain:

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \mathbf{n} = \frac{\sigma}{\epsilon_0}. \quad (1.68)$$

Therefore, the normal component of the electric field is discontinuous by amount σ/ϵ_0 across the charges surface. Note that eq.(1.68) does not determine \mathbf{E}_1 and \mathbf{E}_2 unless the geometry and form of the surface charge is particularly simple. For example, in case of the infinite plane of constant σ by symmetry the electric field is pointing in the opposite directions and the magnitude of the field is $|\mathbf{E}| = \frac{\sigma}{2\epsilon_0}$.

In contrast, the tangential component of the electric field is always continuous across the boundary of the surface. To show this we apply eq.(1.52), which says that

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0, \quad (1.69)$$

to a rectangular loop shown in Fig.1.8.

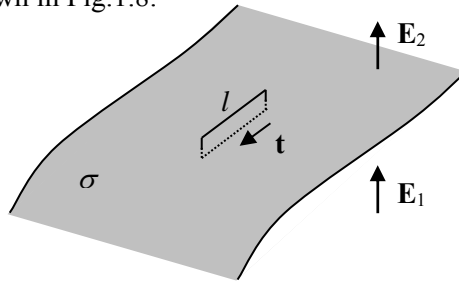


Fig.1.8

If the height of the loop tends to zero the integral is equal to

$$\oint \mathbf{E} \cdot d\mathbf{l} = l(\mathbf{E}_2 - \mathbf{E}_1) \cdot \mathbf{t}, \quad (1.70)$$

where \mathbf{t} is the unit vector parallel to the surface. Note that if σ is inhomogeneous than l should be sufficiently small. Eqs. (1.69) and (1.70) imply that the tangential components of \mathbf{E}_1 and \mathbf{E}_2 must be the equal.

The potential is continuous across any boundary. Indeed, the potential difference is given by

$$V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{l}, \quad (1.71)$$

where A and B are point above and below the surface. As the path length shrinks to zero, so too does the integral, which implies that there is no discontinuity in the potential when we cross the surface.

Dipole potential

At large distance from a localized charge distribution, it “looks” like a point charge. This follows from eq.(1.47) which says that

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r', \quad (1.72)$$

if we assume that $r \gg r'$:

$$\Phi(\mathbf{r}) \approx \frac{Q}{4\pi\epsilon_0 r}. \quad (1.73)$$

where

$$Q = \int \rho(\mathbf{r}') d^3r'. \quad (1.74)$$

If $Q = 0$, the potential and the field are in general not zero and given by a dipolar term. It can be obtained by expanding $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r} - \nabla \left. \frac{1}{|\mathbf{r}-\mathbf{r}'|} \right|_{r'=0} \cdot \mathbf{r}' = \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3}, \quad (1.75)$$

which gives

$$\Phi_{dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}, \quad (1.76)$$

where \mathbf{p} is the dipole moment

$$\mathbf{p} = \int \rho(\mathbf{r}') \mathbf{r}' d^3 r'. \quad (1.77)$$

Discontinuity of the electrostatic potential at the dipole layer

We have seen the electrostatic potential is continuous when crossing a charged surface. This is not the case for a surface dipole layer. A surface dipole layer can be formed at the interfaces between metals and insulator due to charge transfer and is responsible for the electrostatic potential off-set between the metal and the insulator.

A dipole surface layer can be imagined as being formed by letting the surface S have a surface-charge density $\sigma(\mathbf{r})$ on it, and another surface S' , lying close to S , have an equal and opposite surface-charge density on it at neighboring points, as shown in Fig. 1.9. The dipole-layer distribution of *surface dipole moment density* $D(\mathbf{r})$ is formed by letting S' to approach infinitesimally close to S while the surface-charge density $\sigma(\mathbf{r})$ becomes infinite in such a manner that the product of $\sigma(\mathbf{r})$ and the local separation $d(\mathbf{r})$ of S and S' approaches the limit $D(\mathbf{r})$:

$$D(\mathbf{r}) = \lim_{d(\mathbf{r}) \rightarrow 0} \sigma(\mathbf{r}) d(\mathbf{r}), \quad (1.78)$$

The direction of the dipole moment of the layer is normal to the surface S and in the direction going from negative to positive charge, so that $\mathbf{D}(\mathbf{r}) = \mathbf{n}D(\mathbf{r})$.

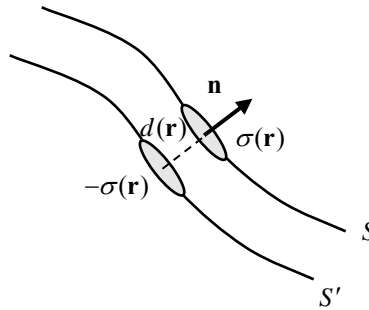


Fig. 1.9 Limiting process for creating a dipole layer.

To find the potential due to a dipole layer we perform mathematically the limiting process described above on the surface-density expression (1.78). In this limit, the dipole moment due to an infinitesimal area element da equals

$$d\mathbf{p}(\mathbf{r}') = \mathbf{n}D(\mathbf{r}') da, \quad (1.79)$$

where \mathbf{n} is the unit normal to the surface S directed away from S' , as shown in Fig. 1.9. According to Eq. (1.76), the potential at point \mathbf{r} caused by this point dipole $d\mathbf{p}$ at point \mathbf{r}' is

$$d\Phi(\mathbf{r}) = \frac{d\mathbf{p}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3} = \frac{D(\mathbf{r}')}{4\pi\epsilon_0} \mathbf{n} \cdot \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} da, \quad (1.80)$$

Equation (1.80) has a simple geometrical interpretation. We note that

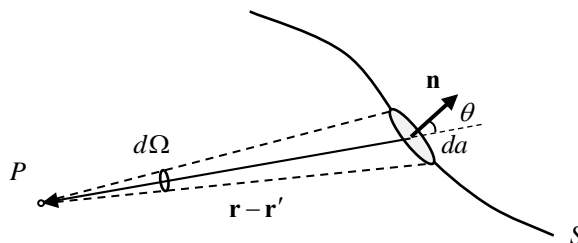
$$\mathbf{n} \cdot \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} da = \mathbf{n} \cdot \frac{1}{|\mathbf{r} - \mathbf{r}'|^2} \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} da = -\frac{\cos\theta da}{|\mathbf{r} - \mathbf{r}'|^2} = -d\Omega, \quad (1.81)$$

where $d\Omega$ is the element of solid angle subtended at the observation point by the area element da , as indicated in Fig. 1.10. Note that $d\Omega$ has a positive sign if θ an acute angle (i.e., when the observation point views the "inner" side of the dipole layer). The potential can be written

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \int_S D(\mathbf{r}') d\Omega. \quad (1.82)$$

For a constant surface-dipole-moment density D , the potential is just the product of the moment (divided by $4\pi\epsilon_0$) and the solid angle subtended at the observation point by the surface, regardless of its shape.

Fig. 1.10 The potential at P due to the dipole layer D on the area element da is just the negative product of D and the solid angle element $d\Omega$ subtended by da at P .



There is a discontinuity in potential in crossing a double layer. This can be seen by letting the observation point come infinitesimally close to the double layer. The double layer is now imagined to consist of two parts, one being a small disc directly under the observation point. The disc is sufficiently small that it is sensibly flat and has constant surface-dipole-moment density D . Evidently the total potential can be obtained by linear superposition of the potential of the disc and that of the remainder. From (1.82) it is clear that the potential of the disc alone has a discontinuity of D/ϵ_0 in crossing from the inner to the outer side, being $-D/2\epsilon_0$ on the inner side and $+D/2\epsilon_0$ on the outer. The potential of the remainder alone, with its hole where the disc fits in, is continuous across the plane of the hole. Consequently, the total potential jump in crossing the surface is:

$$\Phi_2 - \Phi_1 = \frac{D}{\epsilon_0}. \quad (1.83)$$

This result is analogous to the discontinuity of electric field in crossing a surface-charge density. Equation (1.83) can be interpreted "physically" as a potential drop occurring "inside" the dipole layer; it can be calculated as the product of the field between the two layers of surface charge times the separation before the limit is taken.

Electrostatic energy

As follows from Eq. (1.50), the electrostatic energy of a charge in electric field can be defined by

$$U(\mathbf{r}) = q\Phi(\mathbf{r}) , \quad (1.84)$$

where the potential is defined with respect to some reference point, e.g., infinity.

Now we calculate the potential energy of a charge distribution. For this purpose, we need to calculate how much work it takes to assemble an entire collection of N point charges. We bring in charges (q_1, q_2, \dots, q_N), one by one, from far away to their positions ($\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N$). The first charge, q_1 , takes no work, since there is no field yet for perform work against. According to (1.84), bringing in charge q_2 will cost $W_2 = q_2\Phi_1(\mathbf{r}_2)$, where $\Phi_1(\mathbf{r}_2)$ is the potential due to q_1 produced in point \mathbf{r}_2 . Therefore,

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \frac{q_1}{|\mathbf{r}_2 - \mathbf{r}_1|} . \quad (1.85)$$

Bringing q_3 in the position \mathbf{r}_3 in the presence of charges q_1 and q_2 will cost

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left\{ \frac{q_1}{|\mathbf{r}_3 - \mathbf{r}_1|} + \frac{q_2}{|\mathbf{r}_3 - \mathbf{r}_2|} \right\} , \quad (1.86)$$

and so on. Thus, the *total* work necessary to assemble N charges is given by

$$U = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^N \sum_{\substack{j=1 \\ j>i}}^N \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} . \quad (1.87)$$

The stipulation $j > i$ just implies that the same pair of charges should not be counted twice. A more convenient way to represent this is as follows :

$$U = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^N \sum_{\substack{j=1 \\ j\neq i}}^N \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} . \quad (1.88)$$

For a continuous charge distribution this the potential energy takes the form

$$U = \frac{1}{8\pi\epsilon_0} \iint \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r d^3r' . \quad (1.89)$$

where the integrations are unrestricted and include the points $\mathbf{r} = \mathbf{r}'$ because the interaction energy of an infinitesimal continuously distributed charge element with itself vanishes in the limit of zero extent. However, if the charge distribution contains finite point charges, represented by delta functions in $\rho(\mathbf{r})$, then one has to omit the interaction of each of these charges with itself, as in the original sum, Eq. (50), in order to obtain a finite result.

This quantity can also be written in a different form. Recalling that $\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'$ eq. (1.89)

becomes

$$U = \frac{1}{2} \int \rho(\mathbf{r})\Phi(\mathbf{r})d^3r . \quad (1.90)$$

Now we rewrite this result in terms of electric field. Using the Gauss's law we have

$$U = \frac{\epsilon_0}{2} \int (\nabla \cdot \mathbf{E})\Phi d^3r . \quad (1.91)$$

Using the identity

$$\nabla \cdot (\mathbf{E}\Phi) = (\nabla \cdot \mathbf{E})\Phi + \mathbf{E} \cdot (\nabla\Phi) , \quad (1.92)$$

we find that

$$U = \frac{\epsilon_0}{2} \left\{ -\int \mathbf{E} \cdot (\nabla\Phi) d^3r + \int \nabla \cdot (\mathbf{E}\Phi) d^3r \right\} . \quad (1.93)$$

Now taking into account that $\nabla\Phi = -\mathbf{E}$ and using the divergence theorem to evaluate the second integral, we find that

$$U = \frac{\epsilon_0}{2} \left\{ \int_V E^2 d^3r + \oint_S \Phi \mathbf{E} \cdot \mathbf{n} da \right\} . \quad (1.94)$$

If we assume that the charge distribution is *localized* and take the integrals in (1.94) over *all space* it is easy to see that the second integral vanishes. This is because at large distances from the charges the potential goes like $1/r$ and the field goes like $1/r^2$, while the surface area grows like r^2 . Therefore, the surface integral goes down like $1/r$ and vanishes if we integrate over all space. Finally, we have

$$U = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 d^3r . \quad (1.95)$$

The quantity

$$u(\mathbf{r}) = \frac{\epsilon_0 E^2(\mathbf{r})}{2} \quad (1.96)$$

can thus be interpreted as the electrostatic energy density.

Conductors and capacitance

Consider now the special case that our electrostatic system consists of a collection of N electrically isolated conductors.

In a conductor, electrons are able to move freely so as to set up a charge distribution. In the presence of an external electric field, a charge distribution is generated under the influence of this field, which itself gives rise to an electrostatic field. Once equilibrium is attained (about 10^{-18} s in a good conductor), no current flows, and thus the electric field \mathbf{E} is zero throughout the body of a conductor. Since the electric field vanishes in a conductor, the potential must be constant throughout its body, i.e., the conductor is an equipotential.

Using Eq.(1.90), we find for N conductors

$$U = \frac{1}{2} \int \rho(\mathbf{r})\Phi(\mathbf{r})d^3r = \frac{1}{2} \sum_{i=1}^N \int_{V_i} \rho(\mathbf{r})\Phi(\mathbf{r})d^3r = \frac{1}{2} \sum_{i=1}^N \Phi_i \int_{V_i} \rho(\mathbf{r})d^3r = \frac{1}{2} \sum_{i=1}^N Q_i \Phi_i . \quad (1.97)$$

where V_i , Q_i and Φ_i are, respectively, the volume, charge, and potential of the i -th conductor. Due to superposition principle the potential is a linear function of charge. Therefore, we can write

$$\Phi_i = \sum_{j=1}^N p_{ij} Q_j , \quad (i = 1, \dots, N) \quad (1.98)$$

The coefficients p_{ij} are independent of the charges; they depend only on the distribution and shapes of the conductors and are called the *coefficients of potential*. To see that this relation is valid, one needs only think of the potentials produced on each conductor by the given charge Q_j on the j -th conductor and

zero charge on all others; then superpose the solutions to each of the problems of this kind. Indeed, for a single conductor the potential which is produced changes linearly with the charge on this conductor because the charge density also changes linearly with the total charge. Therefore, if the total charge is increased by a factor of p the potential is also increased by a factor of p at any point in space. This statement is also valid in the presence of other conductors.

Inversion of Eqs. (1.98) yields the charges Q_i as linear combinations of the potentials Φ_j :

$$Q_i = \sum_{j=1}^N C_{ij} \Phi_j, \quad (i=1, \dots, N) \quad (1.99)$$

The coefficients C_{ij} are called *coefficients of capacitance*; the diagonal elements C_{ii} are more commonly referred to simply as *capacitances* while the off-diagonal coefficients ones C_{ij} are called *coefficients of electrostatic induction* and are not to be confused with the inductances introduced in connection with Faradays's law. The capacitance of a single conductor, C_{ii} , is thus the total charge on that conductor when it is maintained at a unit potential while all other conductors are held at zero potential.

The energy of the system of conductors may be written in terms of potentials and the coefficients C_{ij} as

$$U = \frac{1}{2} \sum_{i=1}^N Q_i \Phi_i = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N C_{ij} \Phi_i \Phi_j. \quad (1.100)$$

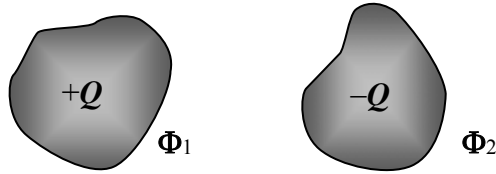


Fig.1.11

Sometimes the capacitance of a system of conductors is also defined. As an example the capacitance of a pair of conductors with equal and opposite charge (Fig.1.11) is defined as the ratio of the charge on one conductor to the potential difference between them when all other conductors are maintained at zero potential.

In this case

$$\begin{pmatrix} Q \\ -Q \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad (1.101)$$

and

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{1}{C_{11}C_{22} - C_{12}C_{21}} \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix} \begin{pmatrix} Q \\ -Q \end{pmatrix}. \quad (1.102)$$

Therefore the capacitance of the two conductors is

$$C = \frac{Q}{|\Phi_1 - \Phi_2|} = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{22} + C_{12} + C_{21}}. \quad (1.103)$$

Capacitance is a purely geometrical quantity, determined by the sizes, shapes, and separation of the two conductors. In SI units, C is measured in *Farads* (F); a Farad is a coulomb-per-volt. Actually, this turns out to be inconveniently large; more practical units are the microfarad (10^{-6} F) and the picofarad (10^{-12} F).

Notice that Φ is, by definition, the potential of the *positive* conductor less that of the negative one; likewise, Q is the charge of *the positive* conductor. Accordingly, capacitance is an intrinsically positive quantity.

Example: Capacitance of a “parallel-plate capacitor” consisting of two metal surfaces of area A held a distance d apart.

If we put $+Q$ on the top and $-Q$ on the bottom, they will spread out uniformly over the two surfaces, provided the area is reasonably large compared to the separation distance. The surface charge density, then, is $\sigma = Q/A$ on the top plate, and so the field between the plates is $E = \sigma / \epsilon_0 = Q / (\epsilon_0 A)$. The potential difference between the plates is therefore $\Phi = Ed = Qd / (\epsilon_0 A)$, and hence $C = Q / \Phi = A\epsilon_0 / d$. If, for instance, the plates are square with sides 1 cm long, and they are held 1 mm apart, then the capacitance is $9 \times 10^{-13} F$.

To “charge up” a capacitor, we have to remove electrons from the positive plate and carry them to the negative plate. In doing so we do work against the electric field, which is pulling them back toward the positive conductor and pushing them away from the negative one. How much work does it take, then, to charge the capacitor up to a final amount Q . Suppose that at some intermediate stage in the process the charge on the positive plate is q , so that the potential difference is q/C . The work we must do to transport the next piece of charge, dq , is $dW = \Phi dq = (q/C) dq$. The total work necessary to charge the capacitor from $q = 0$ to $q = Q$ is

$$W = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C}, \quad (1.104)$$

or since $Q = C\Phi$,

$$W = \frac{1}{2} C \Phi^2, \quad (1.105)$$

where Φ is the final potential on the capacitor. This expression is a particular case of eq.(1.100).

Fields and Forces on Charged Conductors

Another useful application of the expressions for the energy is in the calculation of forces on charged conductors. Consider the surface of a conductor. The normal component of the field at the surface can be inferred from $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$. Consider an integral over a “pillbox” or short circular cylinder oriented with the faces parallel to the surface of a conductor and situated half inside and half outside of the conductor (Fig. 1.12)

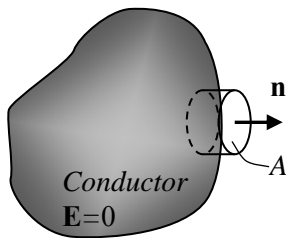


Fig. 1.12

Using the divergence theorem, we may convert to a surface integral.

$$\int_V \nabla \cdot \mathbf{E} d^3r = \oint_S \mathbf{E} \cdot \mathbf{n} da, \quad (1.106)$$

where \mathbf{n} is the normal pointing outward the volume of the pillbox. Given that the height of the cylinder is much smaller than its radius, the only contribution to the surface integral must come from the faces. But $\mathbf{E}(\mathbf{r}) = 0$ on the one inside of the conductor, so we pick up only the contribution from the component of \mathbf{E} normal to the surface of the conductor on the outside. Given that the radius of the cylinder is much smaller than distances over which the field varies appreciably, we obtain

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = \mathbf{E}_n(\mathbf{r})A, \quad (1.107)$$

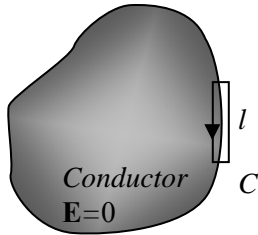
where $\mathbf{E}_n(\mathbf{r})$ is the normal component of the field, A is the area of the face surface and \mathbf{r} is a point just outside of the surface. The volume integral, on the other hand, yields the total charge within the pillbox. Assuming that the height of the pillbox is infinitesimally small, we obtain a vanishingly small contribution from the volume density of charge. Hence, the only contribution comes from the surface charge which is equal to σA . Therefore, according to Gauss' law, we have

$$\mathbf{E}_n(\mathbf{r}) = \frac{\sigma(\mathbf{r})}{\epsilon_0}. \quad (1.108)$$

In order to find the tangential component of the electric field at the surface of a conductor we consider the line integral of $\mathbf{E} \cdot d\mathbf{l}$ around a rectangle which straddles the conductor's surface (Fig.1.13). According to Stock's theorem and the fact that $\nabla \times \mathbf{E} = 0$ this integral must be equal to zero:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0. \quad (1.109)$$

Fig. 1.13



If the the width of the rectangle, which is its size in the direction normal to the interface, is much smaller than its length l , which is its size parallel to the interface, the dominant contribution to the line integral comes from the two sides parallel to the interface. On the inside, $\mathbf{E}(\mathbf{r}) = 0$, so we have only the integral along the side which is exterior to the conductor. Since the whole integral is zero, the integral along this single side must be zero, and hence we can conclude that the tangential component of $\mathbf{E}_t(\mathbf{r})$ just outside of a conductor must vanish:

$$\mathbf{E}_t(\mathbf{r}) = 0. \quad (1.110)$$

Now we are in a position to consider the force on the surface of a conductor. We use the method of virtual work. Imagine moving a small element A of the conductor's surface, along with the charge on it, a distance dx from its initial position in the direction normal to the surface. It will sweep out a volume Adx . To a first approximation, in this volume, we change the electric field from σ/ϵ_0 to zero (since the field is zero within a conductor). This results in the change of the electrostatic energy given by

$$-Adx \frac{\epsilon_0 E^2}{2} = -Adx \frac{\sigma^2}{2\epsilon_0} \quad (1.111)$$

Energy conservation demands that this value is equal to the amount of work dW done on the system in making this displacement. It is also dx times the negative of the electric force acting on the area element dA so that

$$dW = -Fdx = -Adx \frac{\sigma^2}{2\epsilon_0}, \quad (1.112)$$

which implies that

$$\frac{F}{A} = \frac{\sigma^2}{2\epsilon_0}. \quad (1.113)$$

Hence the force per unit area on the surface of the conductor is $\frac{\sigma^2}{2\epsilon_0}$; it is directed normally outward from the conductor (“negative pressure”).