Quantum-Mechanical Analysis of a Longitudinal Stern-Gerlach Electron Beam Splitter

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Abstract

We present the results of a rigorous quantum-mechanical calculation of the propagation of electrons through an inhomogeneous magnetic field with axial symmetry. A complete spin polarization of the beam is demonstrated assuming that a Landau eigenstate can be inserted into the field. This is in contrast with the semi-classical situation, where the spin splitting is blurred. The feasibility of demonstrating such an effect experimentally is considered.

The Stern-Gerlach experiment is one of the most important in the history of physics and is often used to illustrate the nature of spin in quantum mechanics [1]. Curiously, a magnet of the type used by Stern and Gerlach does not work with beams of electrons because of the combined effects of the Lorentz force and the Heisenberg uncertainty principle. This was shown first by Mott and Bohr in 1928 [2]. Subsequently, at the Sixth Solvay Conference on Magnetism, Pauli made a more general argument that no device based on the concept of classical particle trajectories and macroscopic magnetic fields could separate an electron beam by spin[3]. The Bohr/Mott/Pauli argument is codified in numerous textbooks[4].

In this article we show that, in fact, it is possible to observe spin-splitting of a beam of electrons using a *longitudinal* magnetic field configuration instead of the standard transverse geometry of Stern and Gerlach. The longitudinal configuration has the advantage that the electrons experience only off-axis Lorentz forces that are significantly smaller than the on-axis forces in the transverse geometry. Such an idea was first proposed by Brillouin[5], but was specifically rejected by Pauli. Recently, however, we discovered an error in the reasoning Pauli used against Brillouin's idea, and analyzed a counterexample using classical particle trajectories in which spin splitting of electrons could be achieved equal to the blurring caused by the Lorentz forces[6–8]. Although these results are intriguing, they do not take into account the wave nature of the electron, and thus do not address the central question: can a spin separation really be expected? In this article we report the results of a rigorous quantum mechanical analysis of the longitudinal Stern-Gerlach problem, which corresponds to physical reality. We obtain the surprising result that complete separation can be achieved, an improvement over the semiclassical situation. We also consider the problem of producing and inserting an electron wave packet into a real, bounded magnetic field, and find a promising result in this regard as well.

Several previous experiments and quantum-mechanical arguments are of relevance in this context. It is clear that in a non-beam configuration electron spin separation can be effected; Dehmelt and his colleagues have accomplished this with a modified Penning trap[9]. Quantum calculations have also shown spin separations in static situations of this type [10, 11]. For the beam configuration considered by Pauli *et al.*, however, the situation is more ambiguous. Adler [12] and Garroway and Stenholm [13] have shown that the conventional Stern-Gerlach geometry can yield polarization enhancements over narrow spatial regions of the output beam. More importantly for this discussion, Bloch[14] and Dehmelt [9] have sketched quantum arguments for longitudinal field geometries and suggest that complete isolation or filtering of the lowest energy "spin-backward" state should be possible. They did not, however, consider the non-destructive case of full transmission of both spin components in a beam. An experiment of the "Bloch-Dehmelt" type was performed by Knight and his colleagues in the mid 1960s, and observation of a low energy ($< 10^{-8} eV$) tail of (presumably) spin-polarized electrons was observed[15]. This work was never formally published, and, apparently, could not be reproduced [9]. Sannikov [16] and Conte *et al.*[17] considered the problem we take up here, i.e., longitudinal spatial spin separation of a fully transmitted beam, but did not use electron wave packets having extended transverse dimensions. This precludes any elimination of the splitting due to blurring of the separate spin states.

Our calculation begins with the full non-relativistic Hamiltonian for an electron in the magnetic field $\vec{B}(\rho, \phi, z)$ of a simple current ring. The ring lies in the x-y plane, has radius R, and is centered at the origin. Thus

$$\vec{B}(\rho, \phi, z) = -\frac{1}{2}\rho\frac{dB(z)}{dz}[\cos(\phi)\hat{\mathrm{x}} + \sin(\phi)\hat{\mathrm{y}}] + B(z)\hat{\mathrm{z}}, \tag{1}$$

where

$$B(z) = B_0\left(\frac{R}{\sqrt{R^2+z^2}}\right)^3, \tag{2}$$

and $B_0 > 0$ is the field at the origin assuming $\rho/R << 1$. Schrödinger's equation for this problem in cylindrical coordinates about the z-axis is

$$[H_L(z) + H_z]\psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{3}$$

$$H_L(z) = -\frac{\hbar^2}{2m_e}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{L_z^2}{2m_e\rho^2} + \omega_L(z)(L_z + gS_z) + \frac{1}{2}m_e[\omega_L(z)]^2\rho^2, \tag{4}$$

$$H_z = -\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial z^2} + g\frac{d\omega_L}{dz}(xS_x + yS_y), \tag{5}$$

where $\omega_L(z)$ is the Larmor frequency $\left(\frac{|eB(z)|}{2m_e c}\right)$, L_z is the operator for the canonical angular momentum of the electron about the z-axis, S_x, S_y, and S_z are the Cartesian components of the spin operator, m_e is the electron mass, and $g = 2(1 + a_e)$ is the anomalous electron gyromagnetic ratio. Eqs. (4) and (5) bear some discussion. The first two terms of (4) and the first term of (5) correspond to the electron's kinetic energy. In (4), both the fourth term and the part of the third term involving L_z correspond to the classical $-\vec{\mu}_{orb} \cdot \vec{B}$ potential of an electron with orbital angular momentum about z. The latter part of the third term in (4) and the second term in (5) are the equivalent terms for the magnetic potential associated with electron spin. If $\vec{B}$ were uniform, the last term of (5) would vanish, and both m_l and m_s, the quantum numbers associated with L_z and S_z, would separately be good. Generally, for the Hamiltonian of (3), only $m_j = m_l + m_s$ is a good quantum number.

We now construct complete wave packet solutions of the form

$$\psi = \sum_n \sum_{m_l} \sum_{m_s} R_n^{(m_l)}(\rho, \phi) \eta_{m_s} a_{nm_l m_s}(z, t), \tag{6}$$

where η is a spinor, the $a_{nm_lm_s}(z,t)$ are functions that contain all of the explicit z and t dependence of ψ as the electrons move along the magnetic field, and

$$H_L R_n^{(m_l)} \eta_{m_s} = E_{nm_lm_s} R_n^{(m_l)} \eta_{m_s}. \tag{7}$$

Note that the $R_n^{(m_l)}(\rho, \phi)$ have a parametric dependence upon z, but otherwise are solutions to the standard Landau problem. They span x-y space and can be written as

$$R_n^{(m_l)}(\rho, \phi) = N_{n|m_l|}(\sqrt{\alpha}\rho)^{|m_l|} L_n^{(|m_l|)}(\alpha\rho^2) \exp(-\frac{1}{2}\alpha\rho^2) e^{im_l\phi}, \tag{8}$$

$$L_n^{(|m_l|)}(x) = \frac{(-1)^n}{n!\, x^{|m_l|}} e^x \frac{d^n}{dx^n}[x^{n+|m_l|} e^{-x}], \tag{9}$$

$$\alpha = \alpha(z) = \frac{m\omega(z)}{\hbar}, \text{ and} \tag{10}$$

$$N_{n|m_l|} = \sqrt{\frac{n!\, \alpha}{\pi(n + |m_l|)!}}, \tag{11}$$

where

$$E_{nm_lm_s} = \hbar\omega_L(z)(2n + |m_l| + m_l + g m_s + 1). \tag{12}$$

In order to leave m_l as an explicit quantum number, we have not followed the common practice of denoting $2n + |m_l|$ as another single integer. The energy levels of the uniform-field Landau Hamiltonian are shown in Fig. 1.

Substituting (6) into (3) and using (7), we obtain a set of coupled partial differential equations for the $\alpha_{nm_l m_s}$. The coupling terms in these equations are crucial because they determine how non-adiabatic the electron transmission process is, i.e., how likely it is for an electron in a given Landau uniform-field eigenstate to depart from that state over the course of its passage through the varying magnetic field. In order to make estimates of these terms, we start with the set of physical quantities considered in the semiclassical calculation of Ref. [6]. The electrons follow a 2m path length that has a midpoint at the center of a 2 cm radius current ring where the field magnitude B_0 is 10T. Their initial speed is taken to be 10^5 m/s. These values provide a reasonable splitting (631 μm) of the spin in a semiclassical model for on-axis trajectories. Moreover, they represent values that are achievable experimentally; 10^5 m/s corresponds to an electron energy of 28 meV, so that patch field effects could be made small [18]. Solenoidal fields comparable to the single-turn ring field considered here are produced routinely. From the point of view of the present work, the 10T maximum field means that magnetic potential energies for the electron along the trajectory will always be $\ll$ 28 meV for the lowest Landau levels.

The above experimental conditions justify two approximations in our solution of the Schrödinger equation for this problem. First, the z-dependence of the Landau Hamiltonian (Eq. 4) could cause transitions between uniform-field Landau eigenstates. We find that these off-diagonal coupling terms can be neglected with one exception: those associated with the magnetic field gradient that arise from the second term on the right of Eq. 5, corresponding to the possibility of electron spin flip. Second, given that the maximum magnetic potential energies are much smaller than the electron kinetic energy, the WKB approximation is valid for the longitudinal wave function propagation.

The spin-flip probability can be characterized by the ratio of coupling terms U:

$$\left|\frac{U_{ij}}{U_{ii}-U_{jj}}\right| = \frac{3(z/R)}{4a_e\sqrt{\alpha_0}R(1+(z/R)^2)^{1/4}}, \tag{13}$$

where i and j label states connected by the transverse spin operator. Although not negligible, this quantity is $< 10^{-2}$ at all values of z, and corresponds to spin-flip probabilities of the same order of magnitude integrated over the full path length. This is also the spin-flipping probability estimated semiclassically in Ref.[6]. Thus the electrons traverse the magnetic field almost completely adiabatically; if a wave packet that corresponds to a Landau eigenfunction can be inserted into the field, the probability is high that it will emerge in the same state at the other end. It is interesting to note that the small amount of spin flipping that does occur is inversely proportional to the electron g-factor anomaly.

We now consider the transmission of the two Landau wave packets with $n, m_l, m_s = 0, 0, \pm 1/2$. These states are most strongly coupled to the $0, \pm 1, \mp 1/2$ states, respectively (see Fig. 1), but, as discussed above, this coupling is sufficiently small to be neglected. The packets are superpositions of plane waves such that

$$a_{00\pm 1/2}(z, t=0) \equiv a_{\pm}(z, 0) = (2\pi\hbar)^{-1/2} \int \phi(p_z) \exp(ip_z z/\hbar) dp_z \tag{14}$$

where $\phi(p_z)$ is taken to be Gaussian-like with a spatial width along z of χ_0^2 and a momentum spread of $\hbar/\sqrt{2\chi_0}$. At later times,

$$a_{\pm}(z, t) = (2\pi\hbar)^{-1/2} \int \phi(p) \exp\left(i\left[\frac{pz}{\hbar} + \delta_{\pm}(E) - \frac{Et}{\hbar}\right]\right) dp, \tag{15}$$

where we have used the WKB approximation for

$$\delta_{\pm}(E) = 2\sqrt{\alpha_0}\, R\, G\left(\frac{E}{(2 \pm a_e)\hbar\omega_0}\right), \tag{16}$$

and

$$G(x) = \sqrt{x} \int_{-\infty}^{\infty} \left[\left(1 - \frac{1}{x(1+\xi^2)^{3/2}}\right)^{1/2} - 1\right] d\xi. \tag{17}$$

The value of χ_0^2 should be chosen to minimize the spreading of the wave packet along $\hat{\mathbf{z}}$ over the electron flight time l/v_0. This condition yields

$$\chi_{0,min} = \sqrt{\frac{\hbar l}{m_e v_0}} \tag{18}$$

and corresponds to 80 μm in the present case.

The WKB phase shift can be expanded as

$$\delta_{\pm}(p) = \delta_0 + \delta_{1,\pm}(p - p_0) + (1/2)\delta_2(p - p_0)^2 + \cdots \tag{19}$$

and interpreted as follows: the wave packet is displaced by a distance $\hbar\delta_{1,\pm}$ relative to its position in the absence of a magnetic potential, and spreads by an amount corresponding to the normal spreading of a free wave packet at time t plus an extra amount corresponding to an additional time increment, $m_e\hbar\delta_2$.

The results of our calculation for the "spin-forward" and "spin-backward" minimum uncertainty wave packets are shown in Fig. 2. Their most striking feature is the virtually complete separation of the two packets, in marked contrast with our previous calculations[6], in which the packets were barely resolved. In those calculations, "magnetic bottle" forces associated with the mechanical orbital angular momentum of the electron "smeared" both packets by an amount equal to their centroid splitting. In the present situation, the individual wave packet spreading is essentially that which one would observe in a field-free measurement, with a very small additional spreading characterized by δ_2. The lack of spreading, when compared with the semiclassical case, results because *the eigenenergies of the electron wave packet depend not on the mechanical angular momentum but on the canonical angular momentum L_z, which is sharp.* Since the electron wave packet is characterized by a sharp eigenenergy, the only relevant "force" acting on it is proportional to the longitudinal spatial derivative of the magnetic potential (and thus $\partial B/\partial z$)) and is thus sharp as well.

A quantitative measure of the spin splitting can be made using the quantity $F = S/W$, where W is the full-width-half-maximum (FWHM) of one of the two spin components along the z-axis, and S is the splitting distance separating the two spin component centroids. Thus $F = 0$ in the field-free case, unity if the spins are just resolved using Rayleigh's criterion, and $\gg 1$ for "complete" splitting. F is plotted in Fig. 3 as a function of mean electron energy E_0. Wave packets obeying the minimum spreading criterion have widths at the end of a given field-free path-length l proportional to $E_0^{-1/4}$ whereas the spin splitting varies as E_0^{-1}. Thus, even at the lowest energies we consider, magnetic splitting always dominates natural wave packet spreading.

We now address the issue of experimental feasibility, which is essentially determined by the difficulty of inserting the $a_{00\pm1/2}$ states into a realistically terminated magnetic field. Our aim here is not to provide a blueprint for experimental confirmation of our findings, but to demonstrate that no fundamental physics prohibits an observation of spin-splitting. The initial packets must have the spatial dimensions and angular momentum properties of the ground Landau states in a minimum-spreading longitudinal configuration. This means that they must be cylindrically symmetric so that $m_l = 0$. Assuming an energy of 28 meV, minimum spreading requires a longitudinal velocity uncertainty of 1 m/s and a pulse duration of 2 ps. These requirements can be met using laser deflection techniques and a series of preliminary apertures in a low-field region [19]. We assume that the Stern-Gerlach field is surrounded by a high-permeability container with a circular entrance aperture on the beam axis. It can be shown that the aperture's diameter should be $d = \sqrt{20h/eB}$, where B is the field several diameters, d, inside the container. This value of d assures maximum overlap between the $a_{00\pm1/2}$ states and an incident wave whose transverse dimensions are much larger than the aperture. In this case, electrons emerging into the high field region have an 82% probability of being in the $n = 0$ state. The chance of being in a state with $n < 16$ is 95%. After the $n = 0$ state, the two states with the largest probability are those with $n = 2$ (5%) and $n = 3$ (6%). The effect of these "contaminant" $m_l = 0$ states on the transmitted electron spectrum is shown in Fig. 2.

For an apparatus whose typical dimension is 1 m, B will be of the order of a gauss at the aperture ($d = 5\mu$). The size of the aperture is large enough to prevent appreciable diffraction as the electrons enter the high field region; their de Broglie wavelength at this energy is $< 10^{-8}m$. Finally we note that numerical solution of Laplace's equation for the magnetic potential in the presence of a small aperture shows that the field "leaks" out into the low-field region with an exponentially decaying far-field spatial dependence. The integral of $\vec{B} \cdot \vec{dl}$ for a straight-line path through this region is at most 10^{-6} gauss $\cdot\, m$. Since the cyclotron radius of a 28 meV electron in a 1 gauss field is ≈ 6 mm, electron plane waves with this energy experience negligible Lorentz-force distortion prior to the entrance aperture.

We speculate that the most pernicious problem in such experiments would be the non-ideal nature of the collimating apertures, manifesting itself in spurious magnetic field effects and scattering from the aperture boundaries themselves. A better experimental approach may well be one of the type discussed by Conte *et al.* [17], in which the beam to be polarized is that of a synchrotron, and the separation is effected through a series of longitudinal Stern-Gerlach "kicks."

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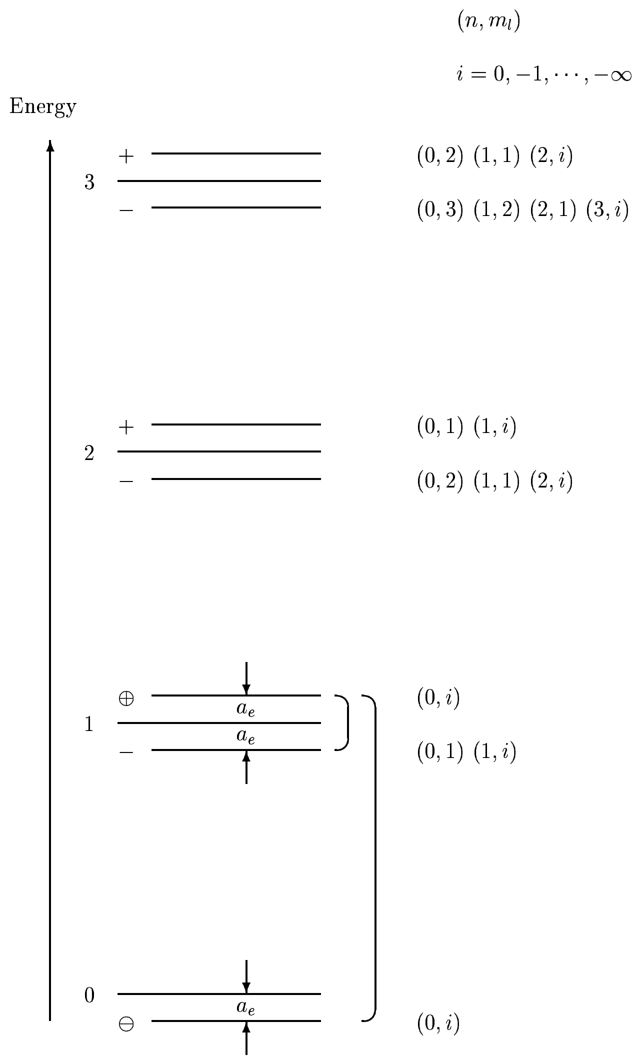

FIG. 1: Energy levels E_{n,m_l,m_s} of the uniform-field Landau Hamiltonian. Minimum uncertainty wave packets (indicated with circles around the sign of m_s) correspond to $n, m_l = 0, 0$, and are most strongly coupled in an inhomogeneous field to the $0, \pm 1$ states, as indicated by the rounded brackets. (see text).

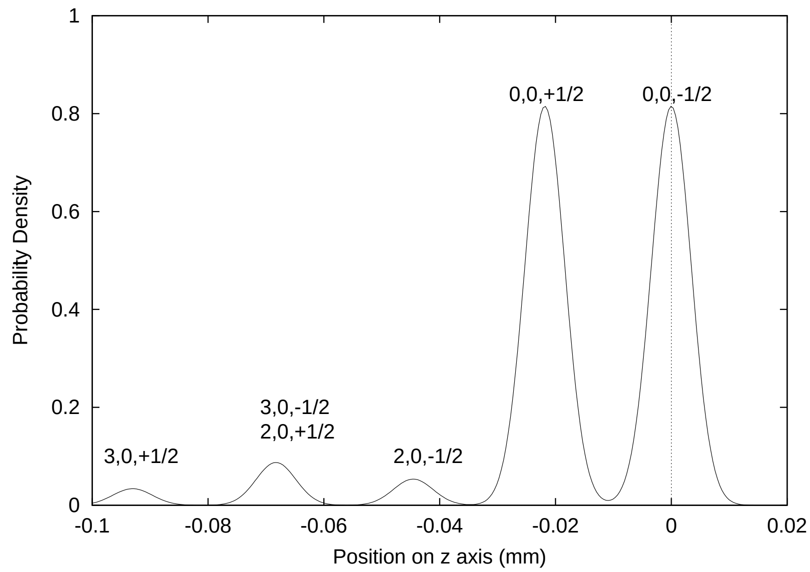

FIG. 2: Electron probability density vs. distance along the z-axis after traversal of a $2m$ flight path. Each peak is marked with its principal (n, m_l, m_s) values. Distance indicated is the deviation from the position of the leading packet, which equals the field-free position. The $n = 2$ and $n = 3$ peaks are "contaminant" contributions caused by the insertion of the electron into the magnetic field (see text).

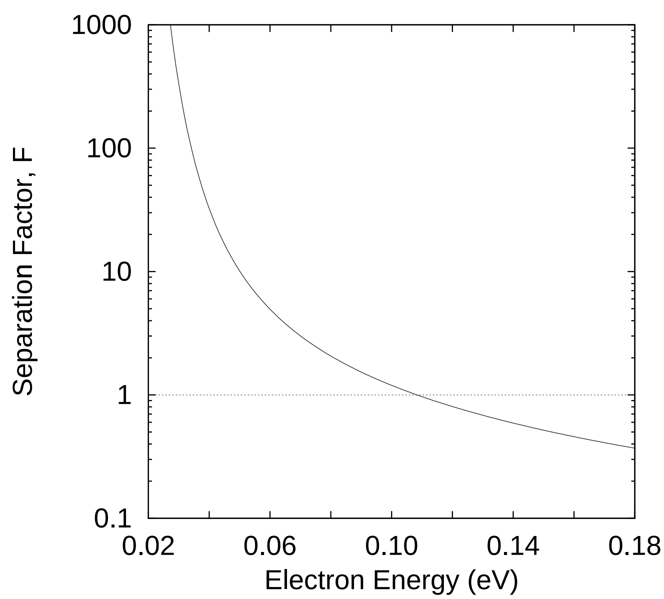

FIG. 3: Spin splitting figure of merit F vs. electron energy. When $F = 1$, the opposite-spin wave packets just meet the Rayleigh resolution criterion.